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Dynamics of entanglement in three coupled harmonic oscillator system with arbitrary time-dependent frequency and coupling constants

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Abstract
The dynamics of mixedness and entanglement is examined by solving the time-dependent Schrödinger equation for three coupled harmonic oscillator system with arbitrary time-dependent frequency and coupling constants parameters. We assume that part of oscillators is inaccessible and remaining oscillators accessible. We compute the dynamics of entanglement between inaccessible and accessible oscillators. In order to show the dynamics pictorially, we introduce three quenched models. In the quenched models, both mixedness and entanglement exhibit oscillatory behavior in time with multi-frequencies. It is shown that the mixedness for the case of one inaccessible oscillator is larger than that for the case of two inaccessible oscillators in the most time interval. Contrary to the mixedness, entanglement for the case of one inaccessible oscillator is smaller than that for the case of two inaccessible oscillators in the most time interval.

Keywords Three coupled harmonic oscillator system · Dynamics of entanglement and mixedness · von Neumann and Rényi entropies

1 Introduction

The most peculiar and counterintuitive properties of quantum mechanics are superposition and entanglement [1–3] of quantum states. In addition to their importance from a pure theoretical aspect, entanglement is known to play a crucial role in the quantum information processing such as quantum teleportation [4], superdense coding [5], quantum cloning [6], quantum cryptography [7,8], and quantum metrology

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[9]. It is also quantum entanglement, which makes the quantum computer outperform the classical one [10,11]. Since quantum technology developed by quantum information processing attracts a considerable attention recently due to limitation of classical technology, it is important to understand the various properties of entanglement.

In the theory of entanglement, the most basic questions are how to detect and how to quantify it from given quantum states. For the last two decades, these questions have been explored mainly in the qubit system. The strategy to the first question is to construct the entanglement witness operators and to explore their properties and applications [12]. The second question has been explored by constructing the various entanglement measures such as distillable entanglement [13], entanglement of formation [13], relative entropy of entanglement [14,15], three-tangle [16,17], *et cetera*.

In spite of construction of many entanglement measures, the analytic computation of these measures is very difficult even in the qubit system[1] except very rare cases. In the real physical system where the quantum state is dependent on continuum variables, computation of such measures is highly difficult or might be impossible. Frequently, thus, we use the von Neumann [18,19] and Rényi entropies [20] to measure the bipartite entanglement of continuum state. Furthermore, the entropies enable us to understand the Hawking–Bekenstein entropy [21–26] of black holes more deeply. They are also important to study on the quantum criticality [27,28] and topological matters [29,30].

In this paper, we will study on the dynamics of entanglement in the three coupled harmonic oscillator system when frequency and coupling constant parameters are arbitrary time-dependent. The harmonic oscillator system is used in many branches of physics due to its mathematical simplicity. The analytical expression of von Neumann entropy was derived for a general real Gaussian density matrix in Ref. [24], and it was generalized to massless scalar field in Ref. [25]. Putting the scalar field system in the spherical box, the author in Ref. [25] has shown that the total entropy of the system is proportional to surface area. This result gives some insights into a question why the Hawking–Bekenstein entropy of black hole is proportional to the area of the event horizon. Recently, the entanglement has been computed in the coupled harmonic oscillator system using a Schmidt decomposition [31]. The von Neumann and Rényi entropies are also explicitly computed in the similar system, called two-site Bose–Hubbard model [32]. More recently, the dynamics of entanglement and uncertainty is exactly derived in the two coupled harmonic oscillator system when frequency and coupling constant parameters are arbitrary time-dependent [33].

In this paper, we assume as follows. Let us consider three coupled harmonic oscillators A, B, and C, whose frequency and coupling constant parameters are arbitrary time-dependent. Let us assume part of oscillator(s) is inaccessible. For example, part of oscillator(s) falls into black hole horizon and as a result, we can access only remaining ones. Under this situation, we derive the time dependence of entanglement between inaccessible and accessible oscillators analytically. As a by-product, we also derive the time dependence of mixedness, which is a trace of square of reduced quantum

[1] However, it is possible to compute entanglement of formation for arbitrary two-qubit state [18,19].

state. If mixedness is one, this means the quantum state is pure. If it is zero, this means the quantum state is completely mixed.

This paper is organized as follows. In the next section, the diagonalization of Hamiltonian is discussed briefly. In Sect. 3, we derive the solutions for time-dependent Schrödinger equation (TDSE) explicitly in the coupled harmonic oscillator system. In Sect. 4, we derive the time dependence of entanglement when A and B oscillators are inaccessible. The time dependence of mixedness for C oscillator is also derived. In Sect. 5, we derive the time dependence of entanglement when A oscillator is inaccessible. The time dependence of mixedness for (B, C)-oscillator system is also derived. In Sect. 6, we introduce three sudden quenched models, where the frequency and coupling constants are abruptly changed at $t = 0$. Using the results of the previous sections, we compare the dynamics of entanglement and mixedness when the inaccessible oscillator(s) is different. In Sect. 7, a brief conclusion is given. In "Appendix A," the quantities α_i, β_i, and γ_{ij}, which appear in the reduced quantum state and have long expressions, are explicitly summarized.

2 Diagonalization of Hamiltonian

The Hamiltonian we will examine in this paper is

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}\Big[K_0(t)(x_1^2 + x_2^2 + x_3^2) + J_{12}(t)(x_1 - x_2)^2 + J_{13}(t)(x_1 - x_3)^2 + J_{23}(t)(x_2 - x_3)^2\Big], \tag{2.1}$$

where $\{x_i, p_i\}$ $(i = 1, 2, 3)$ are the canonical coordinates and momenta. We assume that the frequency parameter K_0 and coupling constants J_{ij} are arbitrarily time-dependent. The Hamiltonian can be written in a form

$$H = \frac{1}{2}\sum_{j=1}^{3} p_j^2 + \frac{1}{2}\sum_{i,j=1}^{3} x_i K_{ij}(t) x_j, \tag{2.2}$$

where

$$K(t) = \begin{pmatrix} K_0 + J_{12} + J_{13} & -J_{12} & -J_{13} \\ -J_{12} & K_0 + J_{12} + J_{23} & -J_{23} \\ -J_{13} & -J_{23} & K_0 + J_{13} + J_{23} \end{pmatrix}. \tag{2.3}$$

The eigenvalues of $K(t)$ are $\lambda_1(t) = K_0$ and $\lambda_\pm(t) = K_0 + J_{12} + J_{13} + J_{23} \pm z$, where

$$z(t) = \sqrt{J_{12}^2 + J_{13}^2 + J_{23}^2 - (J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23})}. \tag{2.4}$$

The corresponding normalized eigenvectors are

$$v_1(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_\pm(t) = A_\pm \begin{pmatrix} -J_{12} + J_{23} \mp z \\ J_{12} - J_{13} \pm z \\ J_{13} - J_{23} \end{pmatrix}, \tag{2.5}$$

where

$$A_\pm(t) = \frac{1}{J_{13} - J_{23}} \left(\frac{2z \pm (J_{13} + J_{23} - 2J_{12})}{6z} \right)^{1/2}. \tag{2.6}$$

Since $K(t)$ is symmetric, v_j $(j = 1, \pm)$ are orthonormal to each other. It is worthwhile noting

$$A_+^2 A_-^2 = \frac{1}{12z^2(J_{13} - J_{23})^2}, \tag{2.7}$$

which is frequently used later. Thus, $K(t)$ can be diagonalized as $K(t) = U^t(t) K_D(t)U(t)$, where

$$U(t) = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ A_+(-J_{12} + J_{23} - z) & A_+(J_{12} - J_{13} + z) & A_+(J_{13} - J_{23}) \\ A_-(-J_{12} + J_{23} + z) & A_-(J_{12} - J_{13} - z) & A_-(J_{13} - J_{23}) \end{pmatrix} \tag{2.8}$$

and $K_D(t) = \mathrm{diag}(\lambda_1, \lambda_+, \lambda_-)$.

Now, we introduce new coordinates

$$\begin{pmatrix} y_1 \\ y_+ \\ y_- \end{pmatrix} = U(t) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{2.9}$$

In terms of the new coordinates, the Hamiltonian (2.2) can be diagonalized in a form

$$H = \frac{1}{2}\left[\pi_1^2 + \omega_1^2(t)y_1^2\right] + \frac{1}{2}\left[\pi_+^2 + \omega_+^2(t)y_+^2\right] + \frac{1}{2}\left[\pi_-^2 + \omega_-^2(t)y_-^2\right], \tag{2.10}$$

where π_j are conjugate momenta of y_j and $\omega_j(t) = \sqrt{\lambda_j}$ $(j = 1, \pm)$.

3 Solutions of TDSE

Consider a Hamiltonian of single harmonic oscillator with arbitrarily time-dependent frequency

$$H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2(t)x^2. \tag{3.1}$$

The TDSE of this system was exactly solved in Ref. [34,35]. The linearly independent solutions $\psi_n(x, t)$ $(n = 0, 1, \ldots)$ are expressed in a form

$$\psi_n(x,t) = e^{-iE_n\tau(t)} e^{\frac{i}{2}\left(\frac{\dot{b}}{b}\right)x^2} \phi_n\left(\frac{x}{b}\right), \tag{3.2}$$

where

$$E_n = \left(n + \frac{1}{2}\right)\omega(0) \quad \tau(t) = \int_0^t \frac{ds}{b^2(s)}$$
$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\omega(0)}{\pi b^2}\right)^{1/4} H_n\left(\sqrt{\omega(0)}x\right) e^{-\frac{\omega(0)}{2}x^2}. \tag{3.3}$$

In Eq. (3.3), $H_n(z)$ is nth-order Hermite polynomial and $b(t)$ satisfies the Ermakov equation

$$\ddot{b} + \omega^2(t)b = \frac{\omega^2(0)}{b^3} \tag{3.4}$$

with $b(0) = 1$ and $\dot{b}(0) = 0$. Solution of the Ermakov equation was discussed in Ref. [36]. If $\omega(t)$ is time-independent, $b(t)$ is simply one. If $\omega(t)$ is instantly changed as

$$\omega(t) = \begin{cases} \omega_i & t = 0 \\ \omega_f & t > 0, \end{cases} \tag{3.5}$$

then $b(t)$ becomes

$$b(t) = \sqrt{\frac{\omega_f^2 - \omega_i^2}{2\omega_f^2}\cos(2\omega_f t) + \frac{\omega_f^2 + \omega_i^2}{2\omega_f^2}}. \tag{3.6}$$

For more general time-dependent case, the Ermakov equation should be solved numerically. Recently, the solution (3.6) is extensively used in Ref. [32] to discuss the dynamics of entanglement for the sudden quenched states of two-site Bose–Hubbard model. Since TDSE is a linear differential equation, the general solution of TDSE is $\Psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x,t)$ with $\sum_{n=0}^{\infty} |c_n|^2 = 1$. The coefficient c_n is determined by making use of the initial conditions.

Using Eqs. (2.10) and (3.2), the general solution for TDSE of the three coupled harmonic oscillators is $\Psi(x_1, x_2, x_3 : t) = \sum_{n_1}\sum_{n_+}\sum_{n_-} c_{n_1,n_+,n_-}\psi_{n_1,n_+,n_-}(x_1, x_2, x_3 : t)$, where $\sum_{n_1}\sum_{n_+}\sum_{n_-} |c_{n_1,n_+,n_-}|^2 = 1$. In terms of y_j given in Eq. (2.9), $\psi_{n_1,n_+,n_-}(x_1, x_2, x_3 : t)$ is expressed as

$$\begin{aligned}\psi_{n_1,n_+,n_-}(x_1, x_2, x_3 : t) &= \frac{1}{\sqrt{2^{n_1+n_++n_-}n_1!n_+!n_-!}}\left(\frac{\omega_1'\omega_+'\omega_-'}{\pi^3}\right)^{1/4} \\ &\times e^{-i[E_{n,1}\tau_1(t)+E_{n,+}\tau_+(t)+E_{n,-}\tau_-(t)]} e^{\frac{i}{2}\left[\left(\frac{\dot{b}_1}{b_1}\right)y_1^2+\left(\frac{\dot{b}_+}{b_+}\right)y_+^2+\left(\frac{\dot{b}_-}{b_-}\right)y_-^2\right]} \\ &\times H_{n_1}\left(\sqrt{\omega_1'}y_1\right) H_{n_+}\left(\sqrt{\omega_+'}y_+\right) H_{n_-}\left(\sqrt{\omega_-'}y_-\right) e^{-\frac{1}{2}\left[\omega_1' y_1^2+\omega_+' y_+^2+\omega_-' y_-^2\right]},\end{aligned} \tag{3.7}$$

where

$$\omega'_j(t) = \frac{\omega_j(0)}{b_j^2} \quad E_{n,j} = \left(n_j + \frac{1}{2}\right)\omega_j(0) \quad \tau_j(t) = \int_0^t \frac{\mathrm{d}s}{b_j^2(s)} \tag{3.8}$$

with $j = 1, \pm$. The scale factors $b_j(t)$ satisfy their own Ermakov equations:

$$\ddot{b}_j + \omega_j^2(t) b_j = \frac{\omega_j^2(0)}{b_j^3} \qquad (j = 1, \pm) \tag{3.9}$$

with $b_j(0) = 1$ and $\dot{b}_j(0) = 0$.

In this paper, we consider only the vacuum solution $\Psi_0(x_1, x_2, x_3 : t) = \psi_{0,0,0}(x_1, x_2, x_3 : t)$. Then, the density matrix of the whole system is given by

$$\begin{aligned} \rho_{ABC}(x_j : x'_j : t) &\equiv \Psi(x_j : t)\Psi^*(x'_j : t) \\ &= \left(\frac{\omega'_1 \omega'_+ \omega'_-}{\pi^3}\right)^{1/2} \exp\left[-\sum_{i,j=1}^{3}\left(x_i G_{ij} x_j + x'_i G^*_{ij} x'_j\right)\right] \end{aligned} \tag{3.10}$$

where $G_{ij} = G_{ji}$ with

$$\begin{aligned} G_{11} &= \frac{1}{2}\left[\frac{v_1}{3} + v_+ A_+^2 (-J_{12} + J_{23} - z)^2 + v_- A_-^2 (-J_{12} + J_{23} + z)^2\right] \\ G_{22} &= \frac{1}{2}\left[\frac{v_1}{3} + v_+ A_+^2 (J_{12} - J_{13} + z)^2 + v_- A_-^2 (J_{12} - J_{13} - z)^2\right] \\ G_{33} &= \frac{1}{2}\left[\frac{v_1}{3} + \left(v_+ A_+^2 + v_- A_-^2\right)(J_{13} - J_{23})^2\right] \\ G_{12} &= \frac{1}{2}\left[\frac{v_1}{3} + v_+ A_+^2 (-J_{12} + J_{23} - z)(J_{12} - J_{13} + z)\right. \\ &\quad \left. + v_- A_-^2 (-J_{12} + J_{23} + z)(J_{12} - J_{13} - z)\right] \\ G_{13} &= \frac{1}{2}\left[\frac{v_1}{3} + \left\{v_+ A_+^2 (-J_{12} + J_{23} - z) + v_- A_-^2 (-J_{12} + J_{23} + z)\right\}(J_{13} - J_{23})\right] \\ G_{23} &= \frac{1}{2}\left[\frac{v_1}{3} + \left\{v_+ A_+^2 (J_{12} - J_{13} + z) + v_- A_-^2 (J_{12} - J_{13} - z)\right\}(J_{13} - J_{23})\right]. \end{aligned} \tag{3.11}$$

In Eq. (3.11), $v_j (j = 1, \pm)$ is defined by

$$v_j = \omega'_j - i \frac{\dot{b}_j}{b_j}. \tag{3.12}$$

In the next two sections, we discuss on the mixedness and entanglement of the reduced states $\rho_C^{(red)}$ and $\rho_{BC}^{(red)}$, respectively.

4 Dynamics of entanglement between *AB* and *C* oscillators

In this section, we assume AB oscillators are inaccessible. Then, the effective state for C oscillator is reduced state, which is given by

$$\rho_C^{(red)}(x_3, x_3' : t) = \mathrm{tr}_{AB}\rho_{ABC} \equiv \int \mathrm{d}x_1 \mathrm{d}x_2 \rho_{ABC}(x_1, x_2, x_3 : x_1, x_2, x_3' : t). \quad (4.1)$$

Performing the integration explicitly, one can show directly

$$\rho_C^{(red)}(x, x' : t) = \left(\frac{\omega_1' \omega_+' \omega_-'}{\pi \Omega} \right)^{1/2} \exp\left[-\frac{1}{\Omega} \left\{ (R_1 - iI_1)x^2 + (R_1 + iI_1)x'^2 - 2Yxx' \right\} \right], \quad (4.2)$$

where

$$\begin{aligned}
\Omega &= \frac{1}{3}\left[A_+^2 Z_+^2 \omega_1' \omega_+' + A_-^2 Z_-^2 \omega_1' \omega_-' + \omega_+' \omega_-'\right] \\
Y &= \frac{|v_1|^2}{36}\left(A_+^2 Z_+^2 \omega_+' + A_-^2 Z_-^2 \omega_-'\right) + \frac{(J_{13} - J_{23})^2 \omega_1'}{12}\left(A_+^4 Z_+^2 |v_+|^2 + A_-^4 Z_-^2 |v_-|^2\right) \\
&\quad + z^2 A_+^2 A_-^2 (J_{13} - J_{23})^4 \left(A_+^2 |v_+|^2 \omega_-' + A_-^2 \omega_+' |v_-|^2\right) \\
&\quad + \frac{A_+^2 A_-^2}{6}(J_{13} - J_{23})^2 \Bigg[\frac{1}{2} Z_+ Z_- \omega_1' (v_+ v_-^* + v_+^* v_-) - z Z_+ \omega_+' (v_1 v_-^* + v_1^* v_-) \\
&\quad + z Z_- \omega_-' (v_1 v_+^* + v_1^* v_+)\Bigg] \\
R_1 &= \frac{1}{2}\omega_1' \omega_+' \omega_-' + Y \\
I_1 &= A_+^2 A_-^2 (J_{13} - J_{23})^2 z \left[Z_+ \omega_1' \omega_+' \frac{\dot{b}_-}{b_-} - Z_- \omega_1' \frac{\dot{b}_+}{b_+} \omega_-' + 2z \frac{\dot{b}_1}{b_1} \omega_+' \omega_-'\right]
\end{aligned} \quad (4.3)$$

with $Z_\pm = 2J_{12} - J_{13} - J_{23} \pm 2z$. It is useful to note

$$Z_+ Z_- = -3(J_{13} - J_{23})^2. \quad (4.4)$$

It is easy to show

$$\mathrm{tr}\left[\rho_C^{(red)}\right] \equiv \int \mathrm{d}x \rho_C^{(red)}(x, x : t) = 1. \quad (4.5)$$

This guarantees the probability conservation of the C-oscillator reduced system. Since $\rho_C^{(red)}$ is a reduced state, it is in general mixed state. The mixedness of $\rho_C^{(red)}$ can be measured by

$$\mathrm{tr}\left[\left(\rho_C^{(red)}\right)^2\right] \equiv \int \mathrm{d}x\mathrm{d}x' \rho_C^{(red)}(x, x' : t) \rho_C^{(red)}(x', x : t) = \sqrt{\frac{\omega_1' \omega_+' \omega_-'}{2(R_1 + Y)}}. \qquad (4.6)$$

Thus, if $Y = 0$, $\rho_C^{(red)}$ becomes pure state. It is completely mixed state when $\omega_1' \omega_+' \omega_-' = 0$.

The entanglement of $\rho_C^{(red)}$ can be computed by solving the eigenvalue equation

$$\int \mathrm{d}x' \rho_C^{(red)}(x, x' : t) f_n(x', t) = p_n(t) f_n(x, t). \qquad (4.7)$$

One can show that the normalized eigenfunction is

$$f_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\epsilon}{\pi}\right)^{1/4} H_n(\sqrt{\epsilon} x) \mathrm{e}^{-\frac{\epsilon}{2} x^2 + i \frac{I_1}{\Omega} x^2}, \qquad (4.8)$$

where

$$\epsilon = 2\sqrt{\frac{R_1^2 - Y^2}{\Omega^2}}, \qquad (4.9)$$

and the corresponding eigenvalue is

$$p_n(t) = [1 - \xi(t)] \xi^n(t), \qquad (4.10)$$

where

$$\xi(t) = \frac{Y}{R_1 + \sqrt{R_1^2 - Y^2}}. \qquad (4.11)$$

Thus, Rényi and von Neumann entropies are given by

$$\begin{aligned} S_\alpha^C &\equiv \frac{1}{1 - \alpha} \ln \mathrm{tr}\left[\left(\rho_C^{(red)}\right)^\alpha\right] = \frac{1}{1 - \alpha} \ln \frac{(1 - \xi)^\alpha}{1 - \xi^\alpha} \\ S_{von}^C &= \lim_{\alpha \to 1} S_\alpha^C = -\ln(1 - \xi) - \frac{\xi}{1 - \xi} \ln \xi. \end{aligned} \qquad (4.12)$$

These quantities measure the entanglement between AB oscillators and C oscillator. The numerical analysis of these quantities will be explored later in the quenched models.

5 Dynamics of entanglement between *A* and *BC* oscillators

In this section, we assume only A oscillator is inaccessible. Then, the effective state for BC oscillator is reduced state, which is given by

$$\rho_{BC}^{(red)}(x_2, x_3 : x_2', x_3' : t) = \mathrm{tr}_A \rho_{ABC} \equiv \int \mathrm{d}x_1 \rho_{ABC}(x_1, x_2, x_3 : x_1, x_2', x_3' : t). \tag{5.1}$$

After long and tedious calculation, one can show

$$\rho_{BC}^{(red)}(x_1, x_2 : y_1, y_2 : t) = \left(\frac{\omega_1' \omega_+' \omega_-'}{\pi^2 A} \right)^{1/2} \mathrm{e}^{-\frac{\Gamma}{A}}, \tag{5.2}$$

where

$$\begin{aligned} A &= G_{11} + G_{11}^* = \frac{\omega_1'}{3} + \omega_+' A_+^2 (-J_{12} + J_{23} - z)^2 + \omega_-' A_-^2 (-J_{12} + J_{23} + z)^2 \\ \Gamma &= (\alpha_1 - i\beta_1) x_1^2 + (\alpha_1 + i\beta_1) y_1^2 + (\alpha_2 - i\beta_2) x_2^2 + (\alpha_2 + i\beta_2) y_2^2 \\ &\quad + 2(\alpha_3 - i\beta_3) x_1 x_2 + 2(\alpha_3 + i\beta_3) y_1 y_2 - 2\gamma_{11} x_1 y_1 - 2\gamma_{22} x_2 y_2 \\ &\quad - 2(\alpha_4 - i\beta_4) x_1 y_2 - 2(\alpha_4 + i\beta_4) x_2 y_1. \end{aligned} \tag{5.3}$$

In Γ α_i, β_i, and γ_{ij} are all real quantities and have long expressions. Their explicit expressions are given in "Appendix A." Here, we present several useful formulae

$$\begin{aligned} \alpha_1 - \gamma_{11} &= \frac{A_+^2}{6} Z_+^2 \omega_1' \omega_+' + \frac{A_-^2}{6} Z_-^2 \omega_1' \omega_-' + 2A_+^2 A_-^2 z^2 (J_{13} - J_{23})^2 \omega_+' \omega_-' \\ \alpha_2 - \gamma_{22} &= \frac{A_+^2}{6} Y_+^2 \omega_1' \omega_+' + \frac{A_-^2}{6} Y_-^2 \omega_1' \omega_-' + 2A_+^2 A_-^2 z^2 (J_{13} - J_{23})^2 \omega_+' \omega_-' \\ \alpha_3 - \alpha_4 &= \frac{A_+^2}{6} Y_+ Z_+ \omega_1' \omega_+' + \frac{A_-^2}{6} Y_- Z_- \omega_1' \omega_-' - 2A_+^2 A_-^2 z^2 (J_{13} - J_{23})^2 \omega_+' \omega_-', \end{aligned} \tag{5.4}$$

where $Y_\pm = J_{12} + J_{13} - 2J_{23} \pm z$. Using Eq. (5.4), it is straight to show

$$(\alpha_1 - \gamma_{11})(\alpha_2 - \gamma_{22}) - (\alpha_3 - \alpha_4)^2 = \frac{\omega_1' \omega_+' \omega_-' A}{4}. \tag{5.5}$$

Then, it is easy to show

$$\mathrm{tr}\left[\rho_{BC}^{(red)} \right] \equiv \int \mathrm{d}x_1 dx_2 \rho_{BC}^{(red)}(x_1, x_2 : x_1, x_2 : t) = 1. \tag{5.6}$$

Also, one can compute the measure of the mixedness for $\rho_{BC}^{(red)}$, which is

$$\begin{aligned} \mathrm{tr}\left[\left(\rho_{BC}^{(red)} \right)^2 \right] &\equiv \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_1 \mathrm{d}y_2 \rho_{BC}^{(red)}(x_1, x_2 : y_1, y_2 : t) \rho_{BC}^{(red)}(y_1, y_2, x_1, x_2 : t) \\ &= \frac{\omega_1' \omega_+' \omega_-' A}{4} \sqrt{\frac{\alpha_2^2 - \gamma_{22}^2}{n_1^2 - n_2^2}}, \end{aligned} \tag{5.7}$$

where

$$n_1 = \alpha_1(\alpha_2^2 - \gamma_{22}^2) - \alpha_2(\alpha_3^2 + \alpha_4^2) + 2\gamma_{22}\alpha_3\alpha_4$$
$$n_2 = \gamma_{11}(\alpha_2^2 - \gamma_{22}^2) + \gamma_{22}(\alpha_3^2 + \alpha_4^2) - 2\alpha_2\alpha_3\alpha_4. \tag{5.8}$$

In order to discuss the entanglement between A oscillator and BC oscillator, we should solve the eigenvalue equation

$$\int \mathrm{d}y_1 \mathrm{d}y_2 \rho_{BC}^{(red)}(x_1, x_2 : y_1, y_2 : t) f_{mn}(y_1, y_2 : t) = p_{mn}(t) f_{mn}(x_1, x_2 : t). \tag{5.9}$$

If the oscillator A is accessible, one can compute the Rényi and von Neumann entropies of $\rho_{BC}^{(red)}$ more easily without solving Eq. (5.9) because the total state ρ_{ABC} is pure. From Schmidt decomposition, we know that the eigenvalue spectrum and, hence, entropies of $\rho_{BC}^{(red)}$ are exactly the same with those of $\rho_A^{(red)}$. Since, however, the oscillator A is assumed to be inaccessible, we should compute the entropies of $\rho_{BC}^{(red)}$ by solving Eq. (5.9) directly. For completeness, we compute the Rényi and von Neumann entropies of $\rho_{BC}^{(red)}$ again in "Appendix B" by making use of $\rho_A^{(red)}$.

In order to solve the eigenvalue Eq. (5.9), we define

$$f_{mn}(x_1, x_2 : t) = \mathrm{e}^{\frac{i}{A}(\beta_1 x_1^2 + \beta_2 x_2^2 + 2\beta_3 x_1 x_2)} g_{mn}(x_1, x_2 : t). \tag{5.10}$$

Then, Eq. (5.9) reduces to

$$C_{\mathcal{N}} \mathrm{e}^{-\frac{1}{A}(\alpha_1 x_1^2 + \alpha_2 x_2^2 + 2\alpha_3 x_1 x_2)} \times \int \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{e}^{-\frac{1}{A}(\alpha_1 y_1^2 + \alpha_2 y_2^2 + 2\alpha_3 y_1 y_2 - 2a y_1 - 2b y_2)} g_{mn}(y_1, y_2 : t) = p_{mn}(t) g_{mn}(x_1, x_2 : t), \tag{5.11}$$

where

$$a(t) = \gamma_{11} x_1 + (\alpha_4 + i\beta_4) x_2 \quad b(t) = (\alpha_4 - i\beta_4) x_1 + \gamma_{22} x_2 \tag{5.12}$$

and $C_{\mathcal{N}}$ is a multiplicative constant. From now on, the multiplicative constant will be absorbed into $C_{\mathcal{N}}$ although it is changed due to Jacobian factors. It can be fixed after calculation is complete by making use of Eq. (5.6).

Now, we define new coordinates

$$\tilde{y_1} = \frac{1}{\mathcal{N}}\left[2\alpha_3 y_1 + \{\eta - (\alpha_1 - \alpha_2)\} y_2\right], \ \tilde{y_2} = \frac{1}{\mathcal{N}}\left[-\{\eta - (\alpha_1 - \alpha_2)\} y_1 + 2\alpha_3 y_2\right]$$
$$\tilde{x_1} = \frac{1}{\mathcal{N}}\left[2\alpha_3 x_1 + \{\eta - (\alpha_1 - \alpha_2)\} x_2\right], \tilde{x_2} = \frac{1}{\mathcal{N}}\left[-\{\eta - (\alpha_1 - \alpha_2)\} x_1 + 2\alpha_3 x_2\right], \tag{5.13}$$

where

$$\eta = \sqrt{(\alpha_1 - \alpha_2)^2 + 4\alpha_3^2} \quad \mathcal{N}^2 = 2\eta[\eta - (\alpha_1 - \alpha_2)]. \tag{5.14}$$

Then, the eigenvalue Eq. (5.11) becomes

$$\begin{aligned} &C_{\mathcal{N}} \mathrm{e}^{-\frac{1}{A}\left(\eta_+ \tilde{x_1}^2 + \eta_- \tilde{x_2}^2\right)} \\ &\times \int \mathrm{d}\tilde{y_1} \mathrm{d}\tilde{y_2} \mathrm{e}^{-\frac{1}{A}\left(\eta_+ \tilde{y_1}^2 + \eta_- \tilde{y_2}^2 - 2\sum_{i,j=1}^{2} c_{ij} \tilde{x_i} \tilde{y_j}\right)} g_{mn}(\tilde{y_1}, \tilde{y_2} : t) = p_{mn}(t) g_{mn}(\tilde{x_1}, \tilde{x_2} : t), \end{aligned} \tag{5.15}$$

where

$$\eta_\pm = \frac{(\alpha_1 + \alpha_2) \pm \eta}{2} \tag{5.16}$$

and

$$\begin{aligned} c_{11} &= \frac{1}{\mathcal{N}^2}\left[4\alpha_3^2 \gamma_{11} + 4\alpha_3\alpha_4 \{\eta - (\alpha_1 - \alpha_2)\} + \gamma_{22} \{\eta - (\alpha_1 - \alpha_2)\}^2\right] \\ c_{22} &= \frac{1}{\mathcal{N}^2}\left[4\alpha_3^2 \gamma_{22} - 4\alpha_3\alpha_4 \{\eta - (\alpha_1 - \alpha_2)\} + \gamma_{11} \{\eta - (\alpha_1 - \alpha_2)\}^2\right] \\ c_{12} &= \frac{1}{\mathcal{N}^2}\left[4\alpha_3^2 \alpha_4 - 2\alpha_3(\gamma_{11} - \gamma_{22}) \{\eta - (\alpha_1 - \alpha_2)\} - \alpha_4 \{\eta - (\alpha_1 - \alpha_2)\}^2 - i\beta_4 \mathcal{N}^2\right] \end{aligned} \tag{5.17}$$

with $c_{21} = c_{12}^*$. In order to simplify Eq. (5.15) some more, we define new coordinates again as

$$\begin{aligned} \bar{x_1} &= \sqrt{\eta_+}\tilde{x_1} \quad \bar{x_2} = \sqrt{\eta_-}\tilde{x_2} \\ \bar{y_1} &= \sqrt{\eta_+}\tilde{y_1} \quad \bar{y_2} = \sqrt{\eta_-}\tilde{y_2}. \end{aligned} \tag{5.18}$$

Then, Eq. (5.15) becomes

$$\begin{aligned} &C_{\mathcal{N}} \mathrm{e}^{-\frac{1}{A}(\bar{x_1}^2 + \bar{x_2}^2)} \int \mathrm{d}\bar{y_1} \mathrm{d}\bar{y_2} \mathrm{e}^{-\frac{1}{A}\left(\bar{y_1}^2 + \bar{y_2}^2 - 2\sum_{i,j=1}^{2} \kappa_{ij} \bar{x_i} \bar{y_j}\right)} g_{mn}(\bar{y_1}, \bar{y_2} : t) \\ &= p_{mn}(t) g_{mn}(\bar{x_1}, \bar{x_2} : t), \end{aligned} \tag{5.19}$$

where

$$\kappa_{11} = \frac{c_{11}}{\eta_+} \quad \kappa_{22} = \frac{c_{22}}{\eta_-} \quad \kappa_{12} = \frac{c_{12}}{\sqrt{\eta_+\eta_-}} \quad \kappa_{21} = \frac{c_{21}}{\sqrt{\eta_+\eta_-}}. \tag{5.20}$$

Since κ_{ij} is a Hermitian matrix, it can be diagonalized by introducing an appropriate unitary matrix. Using the unitary matrix, we define new coordinates finally as

$$\begin{aligned}
X_1 &= \frac{1}{\mathcal{N}_\kappa}\left[2\kappa_{21}\bar{x_1} + \{\chi - (\kappa_{11} - \kappa_{22})\}\bar{x_2}\right] \\
X_2 &= \frac{1}{\mathcal{N}_\kappa}\left[-\{\chi - (\kappa_{11} - \kappa_{22})\}\bar{x_1} + 2\kappa_{12}\bar{x_2}\right] \\
Y_1 &= \frac{1}{\mathcal{N}_\kappa}\left[2\kappa_{21}\bar{y_1} + \{\chi - (\kappa_{11} - \kappa_{22})\}\bar{y_2}\right] \\
Y_2 &= \frac{1}{\mathcal{N}_\kappa}\left[-\{\chi - (\kappa_{11} - \kappa_{22})\}\bar{y_1} + 2\kappa_{12}\bar{y_2}\right],
\end{aligned} \tag{5.21}$$

where

$$\chi = \sqrt{(\kappa_{11} - \kappa_{22})^2 + 4|\kappa_{12}|^2} \quad \mathcal{N}_\kappa^2 = 2\chi\left[\chi - (\kappa_{11} - \kappa_{22})\right]. \tag{5.22}$$

In terms of the new coordinates, Eq. (5.19) is simplified as

$$\begin{aligned}
&C_{\mathcal{N}} e^{-\frac{1}{A}(X_1^2 + X_2^2)} \\
&\times \int dY_1 dY_2 e^{-\frac{1}{A}\left[Y_1^2 + Y_2^2 - 2(\chi_+ X_1 Y_1 + \chi_- X_2 Y_2)\right]} g_{mn}(Y_1, Y_2 : t) = p_{mn}(t) g_{mn}(X_1, X_2 : t),
\end{aligned} \tag{5.23}$$

where

$$\chi_\pm = \frac{1}{2}\left[(\kappa_{11} + \kappa_{22}) \pm \chi\right]. \tag{5.24}$$

Then, Eq. (5.23) is divided into two single variable eigenvalue equations as

$$\begin{aligned}
L_1 e^{-\frac{1}{A}X_1^2} \int dY_1 e^{-\frac{1}{A}(Y_1^2 - 2\chi_+ X_1 Y_1)} g_{1,m}(Y_1, t) &= q_{1,m}(t) g_{1,m}(X_1, t) \\
L_2 e^{-\frac{1}{A}X_2^2} \int dY_2 e^{-\frac{1}{A}(Y_2^2 - 2\chi_- X_2 Y_2)} g_{2,n}(Y_2, t) &= q_{2,n}(t) g_{2,n}(X_2, t),
\end{aligned} \tag{5.25}$$

where

$$\begin{aligned}
L_1 L_1 = C_{\mathcal{N}} \quad p_{mn}(t) &= q_{1,m}(t) q_{2,n}(t) \\
g_{mn}(X_1, X_2 : t) &= g_{1,m}(X_1, t) g_{2.n}(X_2, t).
\end{aligned} \tag{5.26}$$

Each eigenvalue equation in Eq. (5.25) can be solved easily. Then, the normalized eigenfunction of $\rho_{BC}^{(red)}$ is

$$\begin{aligned}
&g_{mn}(X_1, X_2 : t) \\
&= \left[\frac{1}{\sqrt{2^m m!}}\left(\frac{\epsilon_1}{\pi}\right)^{1/4} H_m(\sqrt{\epsilon_1} X_1) e^{-\frac{\epsilon_1}{2}X_1^2}\right]\left[\frac{1}{\sqrt{2^n n!}}\left(\frac{\epsilon_2}{\pi}\right)^{1/4} H_n(\sqrt{\epsilon_2} X_2) e^{-\frac{\epsilon_2}{2}X_2^2}\right]
\end{aligned} \tag{5.27}$$

and the corresponding eigenvalue is

$$p_{mn}(t) = \left[L_1 \sqrt{\frac{\pi}{\frac{1}{A} + \frac{\epsilon_1}{2}}} \left(\frac{\frac{1}{A} - \frac{\epsilon_1}{2}}{\frac{1}{A} + \frac{\epsilon_1}{2}} \right)^{m/2} \right] \left[L_2 \sqrt{\frac{\pi}{\frac{1}{A} + \frac{\epsilon_2}{2}}} \left(\frac{\frac{1}{A} - \frac{\epsilon_2}{2}}{\frac{1}{A} + \frac{\epsilon_2}{2}} \right)^{n/2} \right], \tag{5.28}$$

where

$$\frac{\epsilon_1}{2} = \frac{1}{A}\sqrt{1 - \chi_+^2} \frac{\epsilon_2}{2} = \frac{1}{A}\sqrt{1 - \chi_-^2}. \tag{5.29}$$

Since Eq. (5.6) guarantees $\sum_{mn} p_{m,n}(t) = 1$, one can fix $C_{\mathcal{N}} = L_1 L_2$. Then, $p_{mn}(t)$ becomes

$$p_{mn}(t) = (1 - \xi_1)\xi_1^m (1 - \xi_2)\xi_2^n, \tag{5.30}$$

where

$$\xi_1 = \frac{\chi_+}{1 + \sqrt{1 - \chi_+^2}} \xi_2 = \frac{\chi_-}{1 + \sqrt{1 - \chi_-^2}}. \tag{5.31}$$

Thus, Rényi and von Neumann entropies for $\rho_{BC}^{(red)}$ are given by

$$\begin{aligned} S_\alpha^{BC} &\equiv \frac{1}{1-\alpha} \ln \mathrm{tr} \left[\left(\rho_{BC}^{(red)} \right)^\alpha \right] = S_{1,\alpha} + S_{2,\alpha} \\ S_{von}^{BC} &\equiv \lim_{\alpha \to 1} S_\alpha^{BC} = S_{1,von} + S_{2,von}, \end{aligned} \tag{5.32}$$

where

$$\begin{aligned} S_{1,\alpha} &= \frac{1}{1-\alpha} \ln \frac{(1-\xi_1)^\alpha}{1-\xi_1^\alpha} S_{2,\alpha} = \frac{1}{1-\alpha} \ln \frac{(1-\xi_2)^\alpha}{1-\xi_2^\alpha} \\ S_{1,von} &= -\ln(1-\xi_1) - \frac{\xi_1}{1-\xi_1} \ln \xi_1 \quad S_{2,von} = -\ln(1-\xi_2) - \frac{\xi_2}{1-\xi_2} \ln \xi_2. \end{aligned} \tag{5.33}$$

6 Numerical analysis: sudden quenched models

Using the results of the previous sections, we examine in this section the dynamics of the mixedness and entanglement for $\rho_C^{(red)}$ and $\rho_{BC}^{(red)}$. Although we can consider more general time-dependent cases by solving the Ermakov equation (3.4) numerically, we confine ourselves in this section into the more simple sudden quenched model, where the time dependence of frequency parameter $K_0(t)$ and coupling constants $J_{ij}(t)$ arises from abrupt change at $t = 0$ such as

$$K_0(t) = \begin{cases} K_{0,i} & t = 0 \\ K_{0,f} & t > 0 \end{cases} \qquad J_{ij}(t) = \begin{cases} J_{ij,i} & t = 0 \\ J_{ij,f} & t > 0. \end{cases} \tag{6.1}$$

Then, $\omega_1(t)$ and $\omega_\pm(t)$ defined in the diagonal Hamiltonian (2.10) become

$$\begin{aligned} \omega_{1,i} &= \sqrt{K_{0,i}} \qquad\qquad \omega_{1,f} = \sqrt{K_{0,f}} \\ \omega_{\pm,i} &= \sqrt{K_{0,i} + J_{12,i} + J_{13,i} + J_{23,i} \pm z_i} \\ \omega_{\pm,f} &= \sqrt{K_{0,f} + J_{12,f} + J_{13,f} + J_{23,f} \pm z_f}, \end{aligned} \tag{6.2}$$

where z_i and z_f are initial and later-time values of $z(t)$. Thus, the scale factors $b_\alpha(t)$ $(\alpha = 1, \pm)$ are given by

$$b_\alpha(t) = \sqrt{\frac{\omega_{\alpha,f}^2 - \omega_{\alpha,i}^2}{2\omega_{\alpha,f}^2} \cos\left(2\omega_{\alpha,f} t\right) + \frac{\omega_{\alpha,f}^2 + \omega_{\alpha,i}^2}{2\omega_{\alpha,f}^2}}. \tag{6.3}$$

The trigonometric functions in $b_\alpha(t)$ make oscillatory behavior in the dynamics of mixedness and entanglement.

First, we choose $K_{0,i} = 4$, $K_{0,f} = 6$, $J_{12,i} = 1$, $J_{12,f} = 2$, $J_{13,i} = 3$, $J_{13,f} = 4$, $J_{23,i} = 8$, and $J_{23,f} = 7$. In this case, $\omega_{1,i} = 2$, $\omega_{1,f} = 2.45$, $\omega_{+,i} = 4.72$, $\omega_{+,f} = 4.83$, $\omega_{-,i} = 3.12$, and $\omega_{-,f} = 3.83$. The time dependence of $\text{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ (blue line) and $\text{tr}\left[\left(\rho_{C}^{(red)}\right)^2\right]$ (red line) is plotted in Fig. 1a. As expected, both exhibit oscillatory behavior in time. In the full-time range, $\text{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ is larger than $\text{tr}\left[\left(\rho_{C}^{(red)}\right)^2\right]$. This means $\rho_C^{(red)}$ is more mixed than $\rho_{BC}^{(red)}$. This can be understood as follows. The total state ρ_{ABC} in Eq. (3.10) is pure state. Since $\rho_C^{(red)}$ and $\rho_{BC}^{(red)}$ are effective quantum states when two or one oscillator is lost, respectively, one can expect $\rho_C^{(red)}$ is more mixed than $\rho_{BC}^{(red)}$. Figure 1b shows the time dependence of S_{von}^C (red line) and S_{von}^{BC} (blue line). As expected, both exhibit oscillatory behavior in time due to $b_\alpha(t)$. In the full-time range, S_{von}^C is larger than S_{von}^{BC}. The multi-frequency dependence of von Neumann and Rényi entropies can be seen explicitly if we increases the time domain. Figure 1c, d is time dependence of S_{von}^C and S_{von}^{BC} in $0 \leq t \leq 50$. These figures clearly exhibit the multi-frequency dependence.

Next, we choose time-independent K_0 as $K_0 = 0.1$. Thus, ω_1 is also time-independent as $\omega_1 = 0.316$. The remaining parameters are chosen as $J_{12,i} = 1$, $J_{12,f} = 2$, $J_{13,i} = 2.5$, $J_{13,f} = 3.5$, $J_{23,i} = 3$, and $J_{23,f} = 4$. In this case, $\omega_\pm$ become $\omega_{+,i} = 2.90$, $\omega_{-,i} = 2.19$, $\omega_{+,f} = 3.38$, and $\omega_{-,f} = 2.79$. With these parameters, the dynamics of mixedness and entanglement is plotted in Fig. 2. In Fig. 2a, the

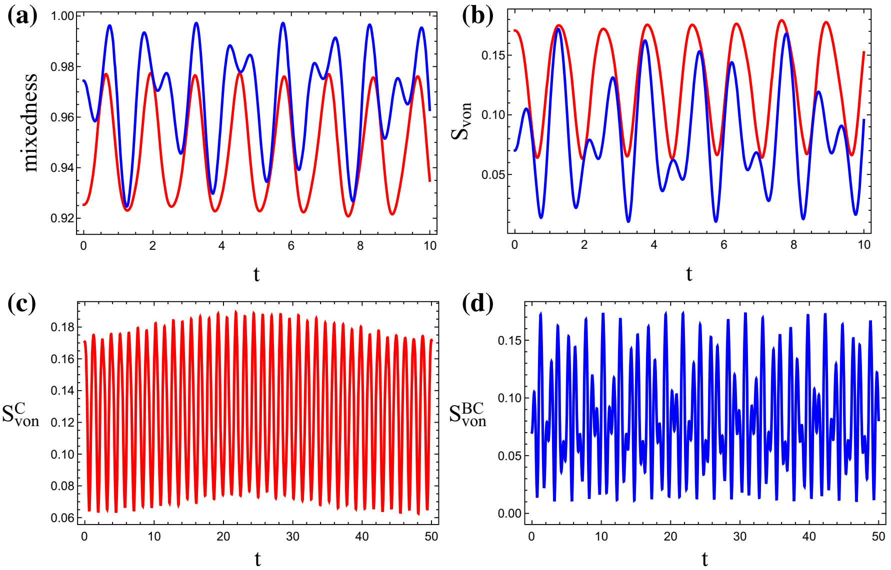

Fig. 1 Time dependence of mixedness (**a**) and von Neumann entropy (**b**) when the quenched parameters are chosen as $K_{0,i} = 4$, $K_{0,f} = 6$, $J_{12,i} = 1$, $J_{12,f} = 2$, $J_{13,i} = 3$, $J_{13,f} = 4$, $J_{23,i} = 8$, and $J_{23,f} = 7$. The red and blue lines correspond to $\rho_C^{(red)}$ and $\rho_{BC}^{(red)}$, respectively. In order to examine the dependence of multi-frequencies, we plot the time dependence of von Neumann entropy for $\rho_C^{(red)}$ (**c**) and $\rho_{BC}^{(red)}$ (**d**) along the long time interval (Color figure online)

time dependence of $\mathrm{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ (blue line) and $\mathrm{tr}\left[\left(\rho_C^{(red)}\right)^2\right]$ (red line) is plotted. Unlike the previous case, $\mathrm{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ is not always larger than $\mathrm{tr}\left[\left(\rho_C^{(red)}\right)^2\right]$ in the full-time range even though the average value of $\mathrm{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ is larger than that of $\mathrm{tr}\left[\left(\rho_C^{(red)}\right)^2\right]$. The time dependence of S_{von}^C (red line) and S_{von}^{BC} (blue line) is plotted in Fig. 2b. Similarly, S_{von}^C is not always larger than S_{von}^{BC} even though it is right in most time interval. In order to examine the effect of constant ω_1, we plot S_{von}^C (Fig. 2c) and S_{von}^{BC} (Fig. 2d) with a long range of time ($0 \leq t \leq 50$). Compared to Fig. 1c, d, the effect of multi-frequency seems to be reduced in Fig. 2c, d.

For completeness, finally, we examine the effect of negative frequency parameter although it is not a physical situation. For this, we choose $K_{0,i} = 0.1$ and $K_{0,f} = -0.1$, which result in $\omega_{1,i} = 0.316$ and $\omega_{1,f} = 0,316i$. The pure imaginary value of $\omega_{1,f}$ changes the cosine factor in $b_1(t)$ into hyperbolic function. Thus, the dynamics of mixedness and entanglement should exhibit oscillatory and exponential behaviors. The remaining parameters are chosen as the same with second example. Then, $\omega_\pm$ become $\omega_{+,i} = 2.90$, $\omega_{-,i} = 2.19$, $\omega_{+,f} = 3,35$, and $\omega_{-,f} = 2.76$. In Fig. 3a, the time dependence of $\mathrm{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ (blue line) and $\mathrm{tr}\left[\left(\rho_C^{(red)}\right)^2\right]$ (red line) is

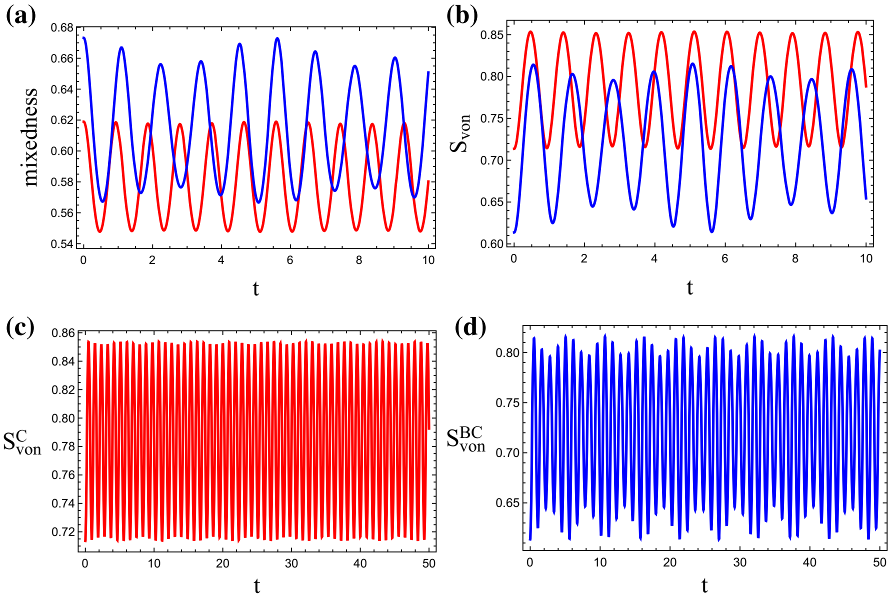

Fig. 2 Time dependence of mixedness (**a**) and von Neumann entropy (**b**) when the quenched parameters are chosen as $K_{0,i} = 0.1$, $K_{0,f} = 0.1$, $J_{12,i} = 1$, $J_{12,f} = 2$, $J_{13,i} = 2.5$, $J_{13,f} = 3.5$, $J_{23,i} = 3$, and $J_{23,f} = 4$. The red and blue lines correspond to $\rho_C^{(red)}$ and $\rho_{BC}^{(red)}$, respectively. In order to examine the dependence of multi-frequencies, we plot the time dependence of von Neumann entropy for $\rho_C^{(red)}$ (**c**) and $\rho_{BC}^{(red)}$ (**d**) along the long time interval. Since constant K_0 gives $b_1(t) = 1$, the effect of multi-frequency seems to be reduced in **c**, **d** compared to Fig. 1**c**, **d** (Color figure online)

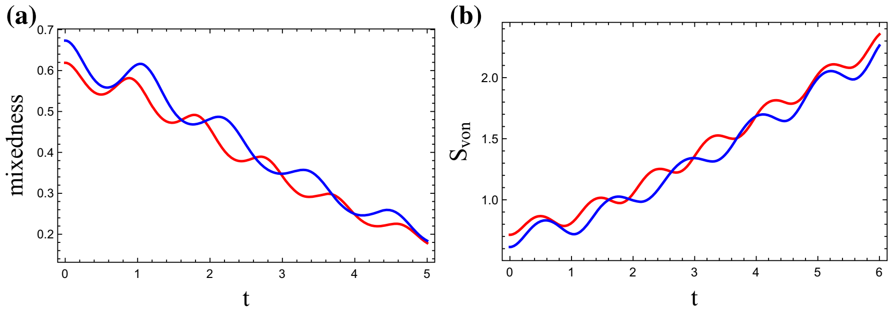

Fig. 3 Time dependence of mixedness (**a**) and von Neumann entropy (**b**) when the quenched parameters are chosen as $K_{0,i} = 0.1$, $K_{0,f} = -0.1$, $J_{12,i} = 1$, $J_{12,f} = 2$, $J_{13,i} = 2.5$, $J_{13,f} = 3.5$, $J_{23,i} = 3$, and $J_{23,f} = 4$. The red and blue lines correspond to $\rho_C^{(red)}$ and $\rho_{BC}^{(red)}$, respectively. Since negative $K_{0,f}$ yields pure imaginary $\omega_{1,f}$, the mixedness and von Neumann entropy exhibit exponential behavior with oscillation generated by ω_+ and ω_- (Color figure online)

plotted. As expected, both exhibit exponential decay with oscillatory behavior. Like the previous models, $\mathrm{tr}\left[\left(\rho_{BC}^{(red)}\right)^2\right]$ is larger than $\mathrm{tr}\left[\left(\rho_{C}^{(red)}\right)^2\right]$ in most time intervals.

In Fig. 3b, the time dependence of S_{von}^{C} (red line) and S_{von}^{BC} (blue line) is plotted. As expected, both also exhibit exponential behavior with oscillation. The unexpected fact is the fact that the von Neumann entropies increase with increasing time. Usually, completely mixed state has zero entanglement in the qubit system. Thus, we expect the decreasing behavior of the von Neumann entropies with increasing time. Figure 3b shows an opposite behavior. Similar behavior can be seen in the two coupled oscillator system with imaginary frequency (see Fig. 2a of Ref. [33]). Probably, this is mainly due to the fact that this third example is unphysical because of negative frequency parameter.

7 Conclusions

The dynamics of mixedness and entanglement is derived analytically by solving the TDSE of the three coupled harmonic oscillator system when the frequency parameter K_0 and coupling constants J_{ij} are arbitrarily time-dependent. For the calculation, we assume that part of oscillator(s) is inaccessible. Thus, we derive the dynamics of entanglement between inaccessible and accessible oscillators. To show the dynamics pictorially, we introduce three sudden quenched models, where Ermakov equation (3.4) can be solved analytically. As expected, due to the scale factors $b_j(t)$, both mixedness and entanglement exhibit oscillatory behavior with multi-frequencies. It is shown that the mixedness for the case of one inaccessible oscillator is larger than that for the case of two inaccessible oscillators in the most time interval. Contrary to the mixedness, entanglement for the case of one inaccessible oscillator is smaller than that for the case of two inaccessible oscillators in the most time interval.

It is natural to extend this paper to n-coupled harmonic oscillator system with arbitrary time-dependent frequency and coupling parameters, whose Hamiltonian can be written as

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\left[K_0(t)\sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} J_{ij}(t)(x_i - x_j)^2\right]. \tag{7.1}$$

Generalizing the method presented in this paper, we think the TDSE of this n-oscillator system can be solved analytically. Assuming that m-oscillator(s) is inaccessible, it seems to be possible to derive the time dependence of entanglement between inaccessible and accessible oscillators. It is of interest to examine the effect of m with fixed n or effect of n with fixed m in the dynamics of entanglement.

Another interesting issue related to this paper is how to compute the tripartite entanglement of the total state (3.10). In qubit system, it is possible to compute the three-tangle for any three-qubit pure state [16]. However, this cannot be directly applied to our realistic system. Probably, we need new computable entanglement measure to explore this issue. We hope to visit this issue in the future.

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Appendix A

The explicit expressions of quantities α_i, β_i, and γ_{ij} in Eq. (5.3) are as follows:

$$
\begin{aligned}
\alpha_1 = & \frac{1}{36}|v_1|^2 + \frac{1}{4}|v_+|^2 A_+^4 (-J_{12} + J_{23} - z)^2 (J_{12} - J_{13} + z)^2 \\
& + \frac{1}{4}|v_-|^2 A_-^4 (-J_{12} + J_{23} + z)^2 (J_{12} - J_{13} - z)^2 \\
& + \frac{A_+^2}{6}\left[Z_+^2 \omega_1' \omega_+' + \left(\omega_1' \omega_+' + \frac{\dot{b_1}}{b_1}\frac{\dot{b_+}}{b_+} \right)(J_{12} - J_{13} + z)(-J_{12} + J_{23} - z) \right] \\
& + \frac{A_-^2}{6}\left[Z_-^2 \omega_1' \omega_-' + \left(\omega_1' \omega_-' + \frac{\dot{b_1}}{b_1}\frac{\dot{b_-}}{b_-} \right)(J_{12} - J_{13} - z)(-J_{12} + J_{23} + z) \right] \\
& + \frac{A_+^2 A_-^2}{2}\Bigg[4z^2 (J_{13} - J_{23})^2 \omega_+' \omega_-' + \left(\omega_+' \omega_-' + \frac{\dot{b_+}}{b_+}\frac{\dot{b_-}}{b_-} \right)(J_{12} - J_{13} + z) \\
& \times (J_{12} - J_{13} - z)(-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z) \Bigg]
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
\beta_1 = & \frac{A_+^2}{6} Z_+ \left[\omega_1' \frac{\dot{b_+}}{b_+}(J_{12} - J_{13} + z) - \frac{\dot{b_1}}{b_1}\omega_+'(-J_{12} + J_{23} - z) \right] \\
& + \frac{A_-^2}{6} Z_- \left[\omega_1' \frac{\dot{b_-}}{b_-}(J_{12} - J_{13} - z) - \frac{\dot{b_1}}{b_1}\omega_-'(-J_{12} + J_{23} + z) \right] \\
& + A_+^2 A_-^2 z (J_{13} - J_{23}) \Bigg[\omega_+' \frac{\dot{b_-}}{b_-}(J_{12} - J_{13} - z)(-J_{12} + J_{23} - z) \\
& - \frac{\dot{b_+}}{b_+}\omega_-'(J_{12} - J_{13} + z)(-J_{12} + J_{23} + z) \Bigg]
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
\alpha_2 = & \frac{1}{36}|v_1|^2 + \frac{1}{4}|v_+|^2 A_+^4 (J_{13} - J_{23})^2 (-J_{12} + J_{23} - z)^2 \\
& + \frac{1}{4}|v_-|^2 A_-^4 (J_{13} - J_{23})^2 (-J_{12} + J_{23} + z)^2 \\
& + \frac{A_+^2}{6}\Bigg[(J_{12} + J_{13} - 2J_{23} + z)^2 \omega_1' \omega_+' \\
& + \left(\omega_1' \omega_+' + \frac{\dot{b_1}}{b_1}\frac{\dot{b_+}}{b_+} \right)(J_{13} - J_{23})(-J_{12} + J_{23} - z) \Bigg] \\
& + \frac{A_-^2}{6}\Bigg[(J_{12} + J_{13} - 2J_{23} - z)^2 \omega_1' \omega_-' \\
& + \left(\omega_1' \omega_-' + \frac{\dot{b_1}}{b_1}\frac{\dot{b_-}}{b_-} \right)(J_{13} - J_{23})(-J_{12} + J_{23} + z) \Bigg] \\
& + \frac{A_+^2 A_-^2}{2}(J_{13} - J_{23})^2 \Bigg[4z^2 \omega_+' \omega_-' \\
& + \left(\omega_+' \omega_-' + \frac{\dot{b_+}}{b_+}\frac{\dot{b_-}}{b_-} \right)(-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z) \Bigg]
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
\beta_2 = {} & \frac{A_+^2}{6}(J_{12} + J_{13} - 2J_{23} + z)\left[\omega_1' \frac{\dot{b}_+}{b_+}(J_{13} - J_{23}) - \frac{\dot{b}_1}{b_1}\omega_+'(-J_{12} + J_{23} - z)\right] \\
& + \frac{A_-^2}{6}(J_{12} + J_{13} - 2J_{23} - z)\left[\omega_1' \frac{\dot{b}_-}{b_-}(J_{13} - J_{23}) - \frac{\dot{b}_1}{b_1}\omega_-'(-J_{12} + J_{23} + z)\right] \\
& - A_+^2 A_-^2 z(J_{13} - J_{23})^2\left[\omega_+' \frac{\dot{b}_-}{b_-}(-J_{12} + J_{23} - z) - \frac{\dot{b}_+}{b_+}\omega_-'(-J_{12} + J_{23} + z)\right]
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
\alpha_3 = {} & \frac{1}{36}|v_1|^2 + \frac{1}{4}|v_+|^2 A_+^4 (J_{13} - J_{23})(J_{12} - J_{13} + z)(-J_{12} + J_{23} - z)^2 \\
& + \frac{1}{4}|v_-|^2 A_-^4 (J_{13} - J_{23})(J_{12} - J_{13} - z)(-J_{12} + J_{23} + z)^2 \\
& + \frac{A_+^2}{12}\Big[2Z_+(J_{12} + J_{13} \\
& - 2J_{23} + z)\omega_1'\omega_+' - \left(\omega_1'\omega_+' + \frac{\dot{b}_1}{b_1}\frac{\dot{b}_+}{b_+}\right)(-J_{12} + J_{23} - z)^2\Big] \\
& + \frac{A_-^2}{12}\Big[2Z_-(J_{12} + J_{13} - 2J_{23} - z)\omega_1'\omega_-' \\
& - \left(\omega_1'\omega_-' + \frac{\dot{b}_1}{b_1}\frac{\dot{b}_-}{b_-}\right)(-J_{12} + J_{23} + z)^2\Big] \\
& + \frac{A_+^2 A_-^2}{2}(J_{13} - J_{23})\Big[-4z^2(J_{13} - J_{23})\omega_+'\omega_-' + \left(\omega_+'\omega_-' + \frac{\dot{b}_+}{b_+}\frac{\dot{b}_-}{b_-}\right)(J_{12} - J_{13}) \\
& \times(-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z)\Big]
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\beta_3 = {} & \frac{A_+^2}{12}\Big[\omega_1' \frac{\dot{b}_+}{b_+}\left\{2(J_{13} - J_{23})(J_{12} - J_{13} + z) + (-J_{12} + J_{23} - z)^2\right\} \\
& + 3\frac{\dot{b}_1}{b_1}\omega_+'(-J_{12} + J_{23} - z)^2\Big] \\
& + \frac{A_-^2}{12}\Big[\omega_1' \frac{\dot{b}_-}{b_-}\left\{2(J_{13} - J_{23})(J_{12} - J_{13} - z) + (-J_{12} + J_{23} + z)^2\right\} \\
& + 3\frac{\dot{b}_1}{b_1}\omega_-'(-J_{12} + J_{23} + z)^2\Big] \\
& + \frac{A_+^2 A_-^2}{2}z(J_{13} - J_{23})\Big[-\omega_+' \frac{\dot{b}_-}{b_-}(-J_{12} + J_{23} - z)(J_{12} - 2J_{13} + J_{23} - z) \\
& + \frac{\dot{b}_+}{b_+}\omega_-'(-J_{12} + J_{23} + z)(J_{12} - 2J_{13} + J_{23} + z)\Big]
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
\alpha_4 = {} & \frac{1}{36}|v_1|^2 + \frac{1}{4}|v_+|^2 A_+^4 (J_{13} - J_{23})(J_{12} - J_{13} + z)(-J_{12} + J_{23} - z)^2 \\
& + \frac{1}{4}|v_-|^2 A_-^4 (J_{13} - J_{23})(J_{12} - J_{13} - z)(-J_{12} + J_{23} + z)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{A_+^2}{12}(-J_{12} + J_{23} - z)^2 \left(\omega_1' \omega_+' + \frac{\dot{b_1}}{b_1} \frac{\dot{b_+}}{b_+} \right) \\
&- \frac{A_-^2}{12}(-J_{12} + J_{23} + z)^2 \left(\omega_1' \omega_-' + \frac{\dot{b_1}}{b_1} \frac{\dot{b_-}}{b_-} \right) \\
&+ \frac{A_+^2 A_-^2}{2}(J_{12} - J_{13})(J_{13} - J_{23})(-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z) \\
&\times \left(\omega_+' \omega_-' + \frac{\dot{b_+}}{b_+} \frac{\dot{b_-}}{b_-} \right)
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
\beta_4 = &\frac{A_+^2}{12}(-J_{12} + J_{23} - z)(J_{12} - 2J_{13} + J_{23} + z) \left(\omega_1' \frac{\dot{b_+}}{b_+} - \frac{\dot{b_1}}{b_1} \omega_+' \right) \\
&+ \frac{A_-^2}{12}(-J_{12} + J_{23} + z)(J_{12} - 2J_{13} + J_{23} - z) \left(\omega_1' \frac{\dot{b_-}}{b_-} - \frac{\dot{b_1}}{b_1} \omega_-' \right) \\
&- \frac{A_+^2 A_-^2}{2} z (J_{13} - J_{23})(-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z) \left(\omega_+' \frac{\dot{b_-}}{b_-} - \frac{\dot{b_+}}{b_+} \omega_-' \right)
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
\gamma_{11} = &\frac{1}{36}|v_1|^2 + \frac{1}{4}|v_+|^2 A_+^4 (J_{12} - J_{13} + z)^2 (-J_{12} + J_{23} - z)^2 \\
&+ \frac{1}{4}|v_-|^2 A_-^4 (J_{12} - J_{13} - z)^2 (-J_{12} + J_{23} + z)^2 \\
&+ \frac{A_+^2}{12}(v_1 v_+^* + v_1^* v_+)(J_{12} - J_{13} + z)(-J_{12} + J_{23} - z) \\
&+ \frac{A_-^2}{12}(v_1 v_-^* + v_1^* v_-)(J_{12} - J_{13} - z)(-J_{12} + J_{23} + z) \\
&+ \frac{A_+^2 A_-^2}{4}(v_+ v_-^* + v_+^* v_-)(J_{12} - J_{13} + z) \\
&\times (J_{12} - J_{13} - z)(-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z)
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
\gamma_{22} = &\frac{1}{36}|v_1|^2 + \frac{1}{4}|v_+|^2 A_+^4 (J_{13} - J_{23})^2 (-J_{12} + J_{23} - z)^2 \\
&+ \frac{1}{4}|v_-|^2 A_-^4 (J_{13} - J_{23})^2 (-J_{12} + J_{23} + z)^2 \\
&+ \frac{A_+^2}{12}(v_1 v_+^* + v_1^* v_+)(J_{13} - J_{23})(-J_{12} + J_{23} - z) \\
&+ \frac{A_-^2}{12}(v_1 v_-^* + v_1^* v_-)(J_{13} - J_{23})(-J_{12} + J_{23} + z) \\
&+ \frac{A_+^2 A_-^2}{4}(v_+ v_-^* + v_+^* v_-)(J_{13} - J_{23})^2 (-J_{12} + J_{23} + z)(-J_{12} + J_{23} - z).
\end{aligned} \tag{A.10}
$$

Appendix B

Since ρ_{ABC} is pure state, it is easy to show that the Rényi and von Neumann entropies of $\rho_{BC}^{(red)}$ are exactly the same with those of $\rho_A^{(red)}$. Thus, we can compute the entropies of $\rho_{BC}^{(red)}$ by solving

$$\int dy \rho_A^{(red)}(x, y : t) g_n(y : t) = q_n(t) g_n(x : t). \tag{B.1}$$

It is straightforward to show that the explicit expression of $\rho_A^{(red)}$ is

$$\begin{aligned}
&\rho_A(red)(x, y : t) \\
&= \int dx_2 dx_3 \rho_{ABC}(x, x_2, x_3 : y, x_2, x_3 : t) \\
&= \left(\frac{\omega_1' \omega_+' \omega_-'}{\pi \Omega_A}\right)^{1/2} \exp\left[-\frac{1}{\Omega_A}\left\{(R_A - iI_A)x^2 + (R_A + iI_A)y^2 - 2Y_A xy\right\}\right],
\end{aligned} \tag{B.2}$$

where

$$\begin{aligned}
\Omega_A &= \frac{1}{3}\left[A_+^2 X_+^2 \omega_1' \omega_+' + A_-^2 X_-^2 \omega_1' \omega_-' + \omega_+' \omega_-'\right] \\
Y_A &= \frac{|v_1|^2}{36}\left(A_+^2 X_+^2 \omega_+' + A_-^2 X_-^2 \omega_-'\right) \\
&\quad + \frac{\omega_1'}{12}\left[A_+^4 X_+^2 (-J_{12} + J_{23} - z)^2 |v_+|^2 + A_-^4 X_-^2 (-J_{12} + J_{23} + z)^2 |v_-|^2\right] \\
&\quad + A_+^2 A_-^2 z^2 (J_{13} - J_{23})^2 \Big[A_+^2 (-J_{12} + J_{23} - z)^2 |v_+|^2 \omega_-' \\
&\quad + A_-^2 (-J_{12} + J_{23} + z)^2 \omega_+' |v_-|^2\Big] \\
&\quad + \frac{A_+^2 A_-^2}{4}(J_{13} - J_{23})\Big[-(J_{12} - J_{13})(-J_{12} + J_{23} + z) \\
&\quad \times (-J_{12} + J_{23} - z)\omega_1'(v_+ v_-^* + v_+^* v_-) \\
&\quad + \frac{2}{3} z X_+ (-J_{12} + J_{23} + z)\omega_+'(v_1 v_-^* + v_1^* v_-) \\
&\quad - \frac{2}{3} z X_- (-J_{12} + J_{23} - z)\omega_-'(v_1 v_+^* + v_1^* v_+)\Big] \\
R_A &= Y_A + \frac{1}{2}\omega_1' \omega_+' \omega_- \\
I_A &= -A_+^2 A_-^2 z (J_{13} - J_{23})\Big[X_+ (-J_{12} + J_{23} + z)\omega_1' \omega_+' \\
&\quad \times \frac{\dot{b}_-}{b_-} - X_- (-J_{12} + J_{23} - z)\omega_1' \frac{\dot{b}_+}{b_+}\omega_-' - 2z(J_{13} - J_{23})\frac{\dot{b}_1}{b_1}\omega_+' \omega_-'\Big]
\end{aligned} \tag{B.3}$$

with $X_{\pm} = J_{12}+J_{23}-2J_{13}\pm z$. It is useful to note $X_+X_- = -3(J_{12}-J_{13})(J_{13}-J_{23})$ and

$$X_{\pm}(-J_{12}+J_{23} \mp z) = -(J_{12} - J_{13})Z_{\pm}X_{\pm}(-J_{12}+J_{23} \pm z) = -(J_{13} - J_{23})W_{\pm}, \tag{B.4}$$

where $Z_{\pm} = 2J_{12} - J_{13} - J_{23} \pm 2z$ and $W_{\pm} = 2J_{23} - J_{12} - J_{13} \pm 2z$.

Following the case of $\rho_C^{(red)}$, it is straightforward to show that the eigenvalue of Eq. (B.1) is $q_n(t) = (1-\xi_A)\xi_A^n$, and the Rényi and von Neumann entropies of $\rho_{BC}^{(red)}$ are given by

$$\begin{aligned} S_{\alpha}^{BC} &\equiv \frac{1}{1-\alpha} \ln \mathrm{tr}\left[\left(\rho_{BC}^{(red)}\right)^{\alpha}\right] = \frac{1}{1-\alpha} \ln \frac{(1-\xi_A)^{\alpha}}{1-\xi_A^{\alpha}} \\ S_{von}^{BC} &= \lim_{\alpha\to 1} S_{\alpha}^{BC} = -\ln(1-\xi_A) - \frac{\xi_A}{1-\xi_A} \ln \xi_A, \end{aligned} \tag{B.5}$$

where

$$\xi_A = \frac{Y_A}{R_A + \sqrt{R_A^2 - Y_A^2}}. \tag{B.6}$$

Although we have not proved analytically that Eqs. (B.5) and (5.32) are exactly the same due to long expressions introduced in "Appendix A," this coincidence is confirmed numerically when plotting Figs. 1d, 2d, and 3b.

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