Three-tangle for rank-three mixed states: Mixture of Greenberger-Horne-Zeilinger, *W*, and flipped-*W* states

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Three-tangle for a rank-3 mixture composed of Greenberger-Horne-Zeilinger, W, and flipped-W states is analytically calculated. The optimal decompositions in the full range of parameter space are constructed by making use of the convex-roof extension. We also provide an analytical technique, which determines whether or not an arbitrary rank-3 state has vanishing three-tangle. This technique is developed by making use of the Bloch sphere S^8 of the qutrit system. The Coffman-Kundu-Wootters inequality is discussed by computing one-tangle and concurrences. It is shown that the one-tangle is always larger than the sum of squared concurrences and three-tangle. The physical implication of three-tangle is briefly discussed.

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Entanglement is a genuine physical resource for the quantum information theories [1]. It is at the heart of the recent much activities on the research of quantum computer.

Entanglement for bipartite mixed states, called concurrence, was studied by Hill and Wootters in Ref. [2] when the density matrix of the state has two or more zero eigenvalues. Subsequently, Wootters extended the result of Ref. [2] to arbitrary bipartite mixed states [3] by making use of the time-reversal operator of the spin-1/2 particle appropriately. In addition, concurrence was used to derive purely tripartite entanglement called residual entanglement or three-tangle [4]. For the mixed three-qubit state ρ the three-tangle is defined by making use of the convex roof construction [5,6] as

$$\tau_3(\rho) = \min \sum_i p_i \tau_3(\rho_i), \tag{1}$$

where the minimum is taken over all possible ensembles of pure states. The ensemble corresponding to the minimum of τ_3 is called optimal decomposition.

Recently, Ref. [7] has shown how to construct the optimal decomposition for the rank-2 mixture of Greenberger-Horne-Zeilinger (GHZ) and W states:

$$\rho(p) = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)|W\rangle\langle W|, \tag{2}$$

where

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \tag{3}$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

The optimal decomposition for $\rho(p)$ was constructed with use of the fact that $\tau_3(|\mathrm{GHZ}\rangle)=1$, $\tau_3(|W\rangle)=0$, and $\langle\mathrm{GHZ}|W\rangle=0$. Once the optimal decompositions are constructed, it is easy to compute the three-tangle. For $\rho(p)$ the three-tangle has three different expressions depending on the range of p as follows:

$$\tau_3(\rho(p)) = \begin{cases} 0 & \text{for } 0 \le p \le p_0, \\ g_I(p) & \text{for } p_0 \le p \le p_1, \\ g_{II}(p) & \text{for } p_1 \le p \le 1, \end{cases} \tag{4}$$

where

$$g_I(p) = p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1-p)^3},$$

$$g_{II}(p) = 1 - (1-p)\left(\frac{3}{2} + \frac{1}{18}\sqrt{465}\right),$$

$$p_0 = \frac{4\sqrt[3]{2}}{3+4\sqrt[3]{2}} \sim 0.6269, \quad p_1 = \frac{1}{2} + \frac{3}{310}\sqrt{465} \sim 0.7087. \tag{5}$$

More recently, this result was extended to the rank-2 mixture of generalized GHZ and generalized W states in Ref. [8].

The purpose of this Brief Report is to extend Ref. [7] to the case of rank-3 mixed states. In this Brief Report we would like to analyze the optimal decompositions for the mixture of GHZ, W, and flipped-W states as

$$\rho(p,q) = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + q|W\rangle\langle W| + (1-p-q)|\tilde{W}\rangle\langle\tilde{W}|, \tag{6}$$

where

$$|\tilde{W}\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle). \tag{7}$$

For simplicity, we will define q as

$$q = \frac{1-p}{n}, \tag{8}$$

where n is a positive integer. Before we go further, it is worthwhile noting that $\rho(p,q)=\rho(p)$ when $n=1$ and, therefore, Eq. (4) is the three-tangle in this case. When $n=\infty$, $\rho(p,q)$ can be constructed from $\rho(p)$ by local-unitary (LU) transformation $\sigma_x \otimes \sigma_x \otimes \sigma_x$. Since the three-tangle is a LU-invariant quantity, the three-tangle of $\rho(p,q)$ with $n=\infty$ is again Eq. (4).

Now we start with three-qubit pure state

$$|Z(p,q,\varphi_1,\varphi_2)\rangle = \sqrt{p}|\text{GHZ}\rangle - e^{i\varphi_1}\sqrt{q}|W\rangle - e^{i\varphi_2}\sqrt{1-p-q}|\widetilde{W}\rangle, \tag{9}$$

whose three-tangle is

$$\tau_3(p,q,\varphi_1,\varphi_2) = \left| p^2 - 4p\sqrt{q(1-p-q)}e^{i(\varphi_1+\varphi_2)} - \frac{4}{3}q(1-p-q)e^{2i(\varphi_1+\varphi_2)} - \frac{8\sqrt{6}}{9}\sqrt{pq^3}e^{3i\varphi_1} - \frac{8\sqrt{6}}{9}\sqrt{p(1-p-q)^3}e^{3i\varphi_2} \right|. \tag{10}$$

The state $|Z(p,q,\varphi_1,\varphi_2)\rangle$ has several interesting properties. First, the mixed state $\rho(p,q)$ in Eq. (8) can be expressed in terms of $|Z(p,q,\varphi_1,\varphi_2)\rangle$ as follows:

$$\rho(p,q) = \frac{1}{3}\left[|Z(p,q,0,0)\rangle\langle Z(p,q,0,0)| + \left|Z\left(p,q,\frac{2\pi}{3},\frac{4\pi}{3}\right)\right\rangle\left\langle Z\left(p,q,\frac{2\pi}{3},\frac{4\pi}{3}\right)\right| + \left|Z\left(p,q,\frac{4\pi}{3},\frac{2\pi}{3}\right)\right\rangle\left\langle Z\left(p,q,\frac{4\pi}{3},\frac{2\pi}{3}\right)\right| \right]. \tag{11}$$

Second, $|Z(p,q,0,0)\rangle$, $|Z(p,q,\frac{2\pi}{3},\frac{4\pi}{3})\rangle$, and $|Z(p,q,\frac{4\pi}{3},\frac{2\pi}{3})\rangle$ have the same three-tangle as shown from Eq. (10) directly. Third, the numerical calculation shows that the p dependence of $\tau_3(p,(1-p)/n,\varphi_1,\varphi_2)$ has many zeros depending on φ_1 and φ_2, but the largest zero $p=p_0$ arises when $\varphi_1=\varphi_2=0$ regardless of n. It can be proven rigorously with use of the implicit function theorem. The n dependence of p_0 is given in Table I. Table I indicates that when n increases from $n=2$, p_0 approaches $4\sqrt[3]{2}/(3+4\sqrt[3]{2}) \sim 0.6269$ from $3/4=0.75$. This is because of the fact that the three-tangle for $\rho(p,q)$ should be Eq. (4) in the $n\to\infty$ limit.

When $p \leqslant p_0$, one can construct the optimal decomposition by making use of Eq. (11) as follows:

$$\rho\left(p,\frac{1-p}{n}\right) = \frac{p}{3p_0}\left[\left|Z\left(p_0,\frac{1-p_0}{n},0,0\right)\right\rangle\left\langle Z\left(p_0,\frac{1-p_0}{n},0,0\right)\right| + \left|Z\left(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\right\rangle\left\langle Z\left(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\right| + \left|Z\left(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\right\rangle\left\langle Z\left(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\right| \right] + \frac{p_0-p}{np_0}|W\rangle\langle W| + \frac{(n-1)(p_0-p)}{np_0}|\widetilde{W}\rangle\langle\widetilde{W}|. \tag{12}$$

Thus, we have vanishing three-tangle in this region:

$$\tau_3\left[\rho\left(p,\frac{1-p}{n}\right)\right] = 0 \quad \text{for } p \leqslant p_0. \tag{13}$$

Now, we consider the $p_0 \leqslant p \leqslant 1$ region. When $p=p_0$, Eq. (12) implies that the optimal decomposition consists of three pure states $|Z(p_0,\frac{1-p_0}{n},0,0)\rangle$, $|Z(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3})\rangle$, and $|Z(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3})\rangle$ with the same probability. This fact together with Eq. (11) strongly suggests that the optimal decomposition at $p_0 \leqslant p$ is described by Eq. (11). As will be shown below, however, this is not true in the full range of $p_0 \leqslant p \leqslant 1$.

The optimal decomposition (11) gives the three-tangle to $\rho(p,q)$ in a form

$$\alpha_I(p) = p^2 - \frac{4\sqrt{n-1}}{n}p(1-p) - \frac{4(n-1)}{3n^2}(1-p)^2 - \frac{8\sqrt{6n}[1+(n-1)^{3/2}]}{9n^2}\sqrt{p(1-p)^3}. \tag{14}$$

Since the three-tangle for the mixed state is defined as a convex roof, $\alpha_I(p)$ should be a convex function if it is a correct three-tangle in the range of $p_0 \leqslant p \leqslant 1$. In order to check this we compute $d^2\alpha_I/dp^2$, which is

$$\frac{d^2\alpha_I(p)}{dp^2} = \frac{2}{9n^2}\left[\{9n^2 + 36n\sqrt{n-1} - 12(n-1)\} - \sqrt{6n}\{1+(n-1)^{3/2}\}\frac{8p^2-4p-1}{\sqrt{p^3(1-p)}} \right]. \tag{15}$$

Using Eq. (15), one can show that $d^2\alpha_I(p)/dp^2 \leqslant 0$ when $p_* \leqslant p \leqslant 1$. The n dependence of p_* is given in Table I. Thus, we need to make $\alpha_I(p)$ convex in the region $p_1 \leqslant p \leqslant 1$, where $p_1 \leqslant p_*$. The constant p_1 will be determined shortly.

For the large-p region one can construct the optimal decomposition as follows:

TABLE I. The n dependence of p_0, p_1, and p_*.

n	1	2	3	10	100	1000
p_0	0.6269	0.75	0.7452	0.712	0.6604	0.6382
p_1	0.7087	0.9330	0.9250	0.8667	0.7710	0.7298
p_*	0.8257	0.9618	0.9572	0.9230	0.8650	0.8391

$$\rho(p,q) = \frac{p-p_1}{1-p_1}|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{1-p}{3(1-p_1)}\left[\left|Z\left(p_1,\frac{1-p_1}{n},0,0\right)\right\rangle\left\langle Z\left(p_1,\frac{1-p_1}{n},0,0\right)\right| + \left|Z\left(p_1,\frac{1-p_1}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\right\rangle \right.$$
$$\left. \times\left\langle Z\left(p_1,\frac{1-p_1}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\right| + \left|Z\left(p_1,\frac{1-p_1}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\right\rangle\left\langle Z\left(p_1,\frac{1-p_1}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\right|\right], \tag{16}$$

which gives the three-tangle in a form

$$\alpha_{II}(p) = \frac{p-p_1}{1-p_1} + \frac{1-p}{1-p_1}\alpha_I(p_1). \tag{17}$$

Note that $d^2\alpha_{II}(p)/dp^2=0$. Thus, $\alpha_{II}(p)$ does not violate the convex constraint of the three-tangle in the large-p region. The parameter p_1 is determined by minimizing $\alpha_{II}(p)$—i.e., $\partial\alpha_{II}/\partial p_1=0$—which gives

$$\frac{4\sqrt{6n}[1+(n-1)^{3/2}]}{9n^2}\frac{2p_1-1}{\sqrt{p_1(1-p_1)}} = 1 + \frac{4\sqrt{n-1}}{n} - \frac{4(n-1)}{3n^2}. \tag{18}$$

The n dependence of p_1 is given in Table I. As expected, p_1 is between p_0 and p_*. When n increases from $n=2$, p_1 decreases from $(2+\sqrt{3})/4\sim 0.933$ to $1/2+3\sqrt{465}/310\sim 0.709$.

In summary, the three-tangle for $\rho(p,q)$ is

$$\tau_3(\rho(p,q)) = \begin{cases} 0 & \text{for } 0\leq p\leq p_0, \\ \alpha_I(p) & \text{for } p_0\leq p\leq p_1, \\ \alpha_{II}(p) & \text{for } p_1\leq p\leq 1, \end{cases} \tag{19}$$

and the corresponding optimal decompositions are (12), (11), and (16), respectively. In order to show that Eq. (19) is genuinely optimal, we plot the p dependence of the three-tangles (10) for various φ_1 and φ_2 when $n=2$ [Fig. 1(a)], $n=3$ [Fig. 1(b)], and $n=10$ [Fig. 1(c)]. These curves have been referred as the characteristic curves [9]. As Ref. [9] indicated, the three-tangle is a convex hull of the minimum of the characteristic curves (thick solid lines in the figure). Figure 1 indicates that the three-tangles (19) plotted as dashed lines are the convex characteristic curves, which implies that Eq. (19) is really optimal.

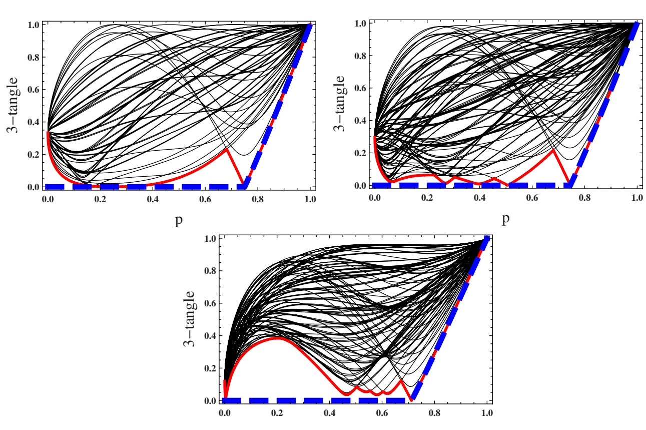

FIG. 1. (Color online) Plot of the p dependence of Eq. (10) for various φ_1 and φ_2. We have chosen φ_1 and φ_2 from 0 to 2π as an interval 0.3. The three figures correspond to $n=2$ (a), $n=3$ (b), and $n=10$ (c), respectively. The minimum curve is plotted as a thick solid line in each figure. These figures indicate that the three-tangle in Eq. (19) (plotted as a dashed line in each figure) is a convex hull of the thick solid line.

The above analysis can be applied to provide an analytical technique which decides whether or not an arbitrary rank-3 state has vanishing three-tangle. First we correspond our states to the qutrit states with

$$|\mathrm{GHZ}\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |W\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |\tilde{W}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{20}$$

It is well known [10] that the density matrix of the arbitrary qutrit state can be represented by $\rho=(1/3)(I+\sqrt{3}\vec{n}\cdot\vec{\lambda})$, where $\vec{n}$ is an eight-dimensional unit vector and $\lambda_i (i=1,\ldots,8)$ are Gell-Mann matrices. Thus the points on the S^8 correspond to pure qutrit states, while the interior points denote mixed states.[1] Then, one can show straightforwardly that the pure states with vanishing three-tangle correspond to

$$|W\rangle \to \left(0,0,-\frac{\sqrt{3}}{2},0,0,0,0,\frac{1}{2}\right),$$

$$|\tilde{W}\rangle \to (0,0,0,0,0,0,0,-1),$$

$$\left|Z\left(p_0,\frac{1-p_0}{n},0,0\right)\right\rangle \to (-\sqrt{3}\xi_1,0,\eta_1,-\sqrt{3}\xi_2,0,\sqrt{3}\xi_3,0,\eta_2),$$

$$\left|Z\left(p_0,\frac{1-p_0}{n},\frac{2\pi}{3},\frac{4\pi}{3}\right)\right\rangle \to \left(\frac{\sqrt{3}}{2}\xi_1,-\frac{3}{2}\xi_1,\eta_1,\frac{\sqrt{3}}{2}\xi_2,\frac{3}{2}\xi_2,\right.$$
$$\left.-\frac{\sqrt{3}}{2}\xi_3,\frac{3}{2}\xi_3,\eta_2\right),$$

$$\left|Z\left(p_0,\frac{1-p_0}{n},\frac{4\pi}{3},\frac{2\pi}{3}\right)\right\rangle \to \left(\frac{\sqrt{3}}{2}\xi_1,\frac{3}{2}\xi_1,\eta_1,\frac{\sqrt{3}}{2}\xi_2,-\frac{3}{2}\xi_2,\right.$$
$$\left.-\frac{\sqrt{3}}{2}\xi_3,-\frac{3}{2}\xi_3,\eta_2\right), \tag{21}$$

where $\xi_1=\sqrt{p_0(1-p_0)/n}$, $\xi_2=\sqrt{n-1}\xi_1$, $\xi_3=\sqrt{n-1}(1-p_0)/n$, $\eta_1=(\sqrt{3}/2)[1-(n+1)(1-p_0)/n]$, and $\eta_2=(1/2)[1-3$

[1] Unlike the qubit system, not all points in S^8 correspond to the qutrit states due to the condition of the star product [10].

$(n-1)(1-p_0)/n]$. Thus these five points form a hyperpolyhedron in eight-dimensional space. Then all rank-3 quantum states corresponding to the points in this hyperpolyhedron have vanishing three-tangle.

Now we would like to consider the Coffman-Kundu-Wootters (CKW) relation [4], which is

$$4 \det \rho_A = \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 + \tau_3(\psi), \tag{22}$$

for the three-qubit pure state $|\psi\rangle$. In Eq. (22), $\mathcal{C}_{AB}$ and $\mathcal{C}_{AC}$ are the concurrences for the corresponding reduced states. Equation (22) indicates that the entanglement of qubit A originates from both bipartite and tripartite contributions. For the mixed state Ref. [4] has shown

$$4 \min[\det(\rho_A)] \geqslant \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2, \tag{23}$$

where the minimum of one-tangle is taken over all possible decompositions of ρ. In Ref. [7] the CKW inequality (23) has been examined for the mixture of GHZ and W states. For this case it was shown that the one-tangle is always larger than the sum of squared concurrences and three-tangle.

Now, we would like to check the CKW inequality for $\rho(p,q)$ in Eq. (6) with $q=(1-p)/n$. In this case one can compute the minimum one-tangle directly, whose expression is

$$\begin{aligned} 4 \min[\det \rho_A] = \frac{1}{9}\{&(8-4p-12q+5p^2+12q^2+12pq) \\ &+ 4\sqrt{pq(1-p-q)}[2\sqrt{6q}+2\sqrt{6(1-p-q)} \\ &- 3\sqrt{p}]\}. \end{aligned} \tag{24}$$

Also it is straightforward to compute the sum of squared concurrences, which is

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 = 2\left(\max\left[0, \tfrac{2}{3}(1-p) - \tfrac{1}{3}\sqrt{(3p+2q)(2+p-2q)}\right]\right)^2. \tag{25}$$

The one-tangle (upper solid lines), $\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2$ (left solid lines), and three-tangle (right solid lines) are plotted in Fig. 2 for $n=1$, $n=2$, and $n=10$. This figure indicates that all quantities approach their corresponding $n=1$ quantity when n increases from $n=2$. This is consistent with the fact that $\rho(p,q)$ with $n=1$ is LU equivalent to $\rho(p,q)$ with $n=\infty$. The inequality

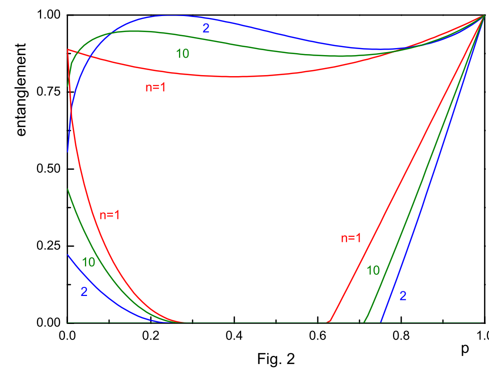

FIG. 2. (Color online) The p dependence of one-tangle (upper solid lines), sum of squared concurrences (left solid lines), and three-tangle (right solid lines) for $n=1$, 2, and 10. This figure clearly indicates that not only the CKW inequality (23), but also (26), holds for all integer n.

$$4 \min[\det(\rho_A)] \geqslant \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 + \tau_3 \tag{26}$$

holds for all n. In the region $p_C \leqslant p \leqslant p_0$, where

$$p_C = \frac{(7n^2-4n+4) - 3n\sqrt{5n^2-4n+4}}{(n-2)^2}, \tag{27}$$

both $\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2$ and τ_3 vanish while there is quite substantial one-tangle. Its interpretation is given in Ref. [7] from the mathematical point of view. However, its physical meaning is still unclear at least for us. In the region $p \geqslant p_C$ and $p \leqslant p_0$ the entanglement of the qubit A mainly stems from the bipartite and tripartite correlations, respectively.

From the point of view of physics it is also of interest to investigate the physical role of the three-tangle. As shown in Ref. [11] two-qubit mixed-state entanglement provides information on the fidelity in bipartite teleportation through noisy channels. Since the three-tangle is purely tripartite entanglement, it may give certain information in the scheme of quantum copy machines or three-party quantum teleportation [12]. It seems to be interesting to explore the physical role of three-tangle in particular real tasks.

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