# Amplitude damping for single-qubit system with single-qubit mixed-state environment

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## Abstract

We study a generalized amplitude damping channel when environment is initially in the single-qubit mixed state. Representing the affine transformation of the generalized amplitude damping by a three-dimensional volume, we plot explicitly the volume occupied by the channels simulatable by a single-qubit mixed-state environment. As expected, this volume is embedded in the total volume by the channels which is simulated by a two-qubit enviroment. The volume ratio is approximately 0.08 which is much smaller than 3/8, the volume ratio for generalized depolarizing channels.

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## 1. Introduction

About three decades ago Feynman [1, 2] suggested that a mathematical computation can be efficiently performed by making use of quantum mechanics. This suggestion seems to be a starting point for the current active research of quantum computer. Ten years later after Feynman's suggestion Shor [3] developed the efficient factoring algorithm for the large integer in the quantum computer. Shor's factoring algorithm makes the most current cryptographic methods useless, when the quantum computer is constructed. Subsequently, the efficient search algorithm was developed by Grover [4, 5]. The factoring and search algorithms were reviewed in [6] from the physically-motivated aspect. Recently, Shor's factoring algorithm was realized in NMR [7] and optical [8] experiments. In addition, the quantum search algorithm was also physically realized in [9, 10].

The quantum computer uses frequently the unitary evolution of the closed quantum system. If, however, the quantum system interacts with environment, the system takes the unwanted non-unitary evolution, which appears as noise in quantum information processing. Therefore, we should understand and control such noise process [11].

In this paper, we would like to study the effect of the environment when the principal system is a single-qubit pure state. In order for the principal system to evolve generally it is well known that we need the two-qubit environment [12]. However, it is argued in [13] that one-qubit mixed-state environment might be sufficient to simulate the most general quantum evolution of a single-qubit system. This argument conjectured in [13] by counting the number of independent parameters.

Later, however, many single-qubit principal channels were found, which cannot be simulated by a single-qubit environment [14, 15]. Furthermore, recently, [16] has shown that only 3/8 of the generalized depolarizing channels can be simulated by the one-qubit mixed-state environment.

In this paper, we would like to extend [16] by examining the amplitude damping channel. The amplitude damping is an important quantum noise, which describes the effect of energy dissipation. The quantum noise is usually explored using a quantum operation $\varepsilon(\rho)$, which is a *convex-linear* map from density operator of the input space to that of the output space, i.e. $\rho_{\text{out}} = \varepsilon(\rho_{\text{in}})$ [11]. In this language the amplitude damping is described via operator-sum representation as

$$\varepsilon_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \tag{1.1}$$

where operation elements $E_0$ and $E_1$ are

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{1.2}$$

and the parameter $\gamma$ represents the probability for energy loss due to losing a particle. Since the density operator of the single-qubit system can be always expressed as $\rho = (\mathbb{1} + \vec{r} \cdot \vec{\sigma})/2$ and $\varepsilon(\rho) = (\mathbb{1} + \vec{s} \cdot \vec{\sigma})/2$ where $\sigma_i$'s are Pauli matrices, the amplitude damping (1.1) can be differently expressed from one Bloch vector $\vec{r}$ to another Bolch vector $\vec{s}$ in the following:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\gamma}{2} \end{pmatrix}. \tag{1.3}$$

The map from $\vec{r}$ to $\vec{s}$ is called affine map and it, in general, is very useful to visualize the effect of quantum operation in Bloch sphere. In this paper, we will generalize the amplitude damping and its corresponding affine map. Making use of the generalized map we will plot explicitly the three-dimensional volume, each point inside of which represents a state which can be reached from a pure initial state when the environment is a two-qubit pure state. The volume is compared with another volume derived from the single-qubit mixed-state environment. It will be shown graphically that the latter volume is embedded in the former, which indicates that the single-qubit mixed-state environment cannot simulate the whole channels derived from the two-qubit environment.

This paper is organized as follows. In section 2, we briefly review [16]. In section 3, the generalized amplitude damping (GAD) is considered. It is shown that the affine map of GAD allows the double-degenerate transformation matrix $M$. It also allows that only the last component of the translation vector $\vec{C}$ is nonvanishing. In section 4, we tried to find the GAD when the environment is a single-qubit mixed state. It is shown by plotting the three-dimensional volume that the GAD channels simulated from the single-qubit environment have very small portion compared to those simulated from the two-qubit environment. The volume ratio is numerically computed and is approximately 0.08, which is much smaller than the ratio 3/8 for generalized depolarizing channels. Section 5 summarizes conclusion and further research direction briefly.
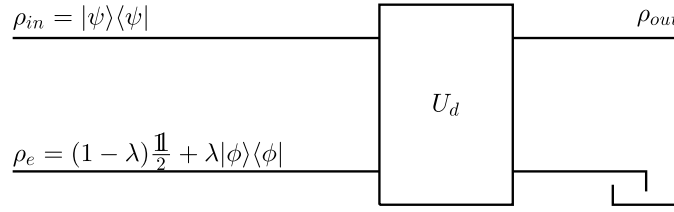
**Figure 1.** Circuit model for the single-qubit channel in the presence of the single-qubit mixed-state environment. The principal system is in $|\psi\rangle$ initially and the environment is in a mixed state $\rho_e$. After unitary interaction via $U_d$, the environment will be traced out.

## 2. Brief review: one-qubit system with one-qubit environment

In this section, we consider a composed closed system which consists of one-qubit principal system and one-qubit mixed-state environment as pictorially depicted in figure 1. Since similar situation was rigorously discussed elsewhere [16], we would like to review it briefly.

We assume the principal system is initially in the pure state, i.e. $\rho_{\text{in}} = |\psi\rangle\langle\psi|$, where

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{-i\phi}\sin\frac{\theta}{2}|1\rangle. \tag{2.1}$$

This state is represented as a point in the Bloch sphere [11].

Next we define the initial state of the environment. In order to control the mixed status of the initial state we introduce a real parameter $\lambda$ and define

$$\rho_e = (1-\lambda)\frac{\mathbb{1}}{2} + \lambda|\phi\rangle\langle\phi| \tag{2.2}$$

where

$$|\phi\rangle = \cos\frac{\xi}{2}|0\rangle + e^{-i\eta}\sin\frac{\xi}{2}|1\rangle. \tag{2.3}$$

Thus $\lambda = 0$ and $\lambda = 1$ correspond to the completely mixed state and pure state, respectively. If $0 < \lambda < 1$, the environment is in the partially mixed state.

Since the joint system is assumed to be closed, the interaction between the physical system and the environment is represented by the unitary matrix $U_d$, which is an element of $SU(4)$. Thus this evolution matrix has generally 15 free parameters. As [17] has shown, however, the number of these free parameters can be reduced to three by making use of the local $SU(2)$ unitary operators. Furthermore, it was shown in the same reference that this three-parameter family of $U_d$ is simply expressed in the Bell basis. Transforming the matrix representation of $U_d$ into the computational basis with discarding the unimportant global phase factor simply yields

$$U_d = \begin{pmatrix} \cos\frac{\alpha+\gamma}{2} & 0 & 0 & i\sin\frac{\alpha+\gamma}{2} \\ 0 & \cos\frac{\alpha-\gamma}{2}e^{-i\beta} & i\sin\frac{\alpha-\gamma}{2}e^{-i\beta} & 0 \\ 0 & i\sin\frac{\alpha-\gamma}{2}e^{-i\beta} & \cos\frac{\alpha-\gamma}{2}e^{-i\beta} & 0 \\ i\sin\frac{\alpha+\gamma}{2} & 0 & 0 & \cos\frac{\alpha+\gamma}{2} \end{pmatrix} \tag{2.4}$$

where $\alpha$, $\beta$ and $\gamma$ are real free parameters.

Since $\rho_{\text{in}}$, $\rho_e$ and $U_d$ are given, $\rho_{\text{out}}$ can be explicitly computed by unitary evolution and partial trace in the following:

$$\rho_{\text{out}} = \text{Tr}_{\text{env}}\left[U_d(\rho_{\text{in}} \otimes \rho_e)U_d^\dagger\right]. \tag{2.5}$$

Let us assume $\rho_{\text{in}} = (\mathbb{1} + \vec{r}\cdot\vec{\sigma})/2$ and $\rho_{\text{out}} = (\mathbb{1} + \vec{r}'\cdot\vec{\sigma})/2$. Then the quantum operation defined as

$$\varepsilon(\rho_{\text{in}}) = \rho_{\text{out}} \tag{2.6}$$

is given by the affine map

$$r_i \rightarrow r_i' = M_{ij} r_j + C_j \tag{2.7}$$

where $M_{ij}$ is $3 \times 3$ real matrix in the form

$$M_{ij} = \begin{pmatrix} \cos\beta\cos\gamma & \lambda\cos\xi\sin\beta\cos\gamma & -\lambda\sin\xi\sin\eta\cos\beta\sin\gamma \\ -\lambda\cos\xi\cos\alpha\sin\beta & \cos\alpha\cos\beta & \lambda\sin\xi\cos\eta\sin\alpha\cos\beta \\ \lambda\sin\xi\sin\eta\cos\alpha\sin\gamma & -\lambda\sin\xi\cos\eta\sin\alpha\cos\gamma & \cos\alpha\cos\gamma \end{pmatrix} \tag{2.8}$$

and the column vector $\vec{C}$ is

$$\vec{C} = -\lambda \begin{pmatrix} \sin\xi\cos\eta\sin\beta\sin\gamma \\ \sin\xi\sin\eta\sin\alpha\sin\beta \\ \cos\xi\sin\alpha\sin\gamma \end{pmatrix}. \tag{2.9}$$

This affine map gives a parametrization of all the channels simulated by a one-qubit mixed-state environment. Varying the six parameters $\alpha, \beta, \gamma, \lambda, \xi$ and $\eta$, we can obtain the various output states $\rho_{\text{out}}$. We can use this various output states to explore the damping effect of the principal system arising due to the interaction with the environment.

For later use we would like to discuss the eigenvalues $\Lambda$ of $M^\dagger M$. To compute $\Lambda$ we should solve the highly complicated third-order equation

$$-\Lambda^3 + f_1 \Lambda^2 + f_2 \Lambda + f_3 = 0 \tag{2.10}$$

where

$$\begin{aligned}
f_1 &= \cos^2\alpha\cos^2\beta + \cos^2\beta\cos^2\gamma + \cos^2\gamma\cos^2\alpha + \lambda^2[\cos^2\xi\sin^2\beta(\cos^2\alpha + \cos^2\gamma) \\
&\quad + \sin^2\xi\sin^2\eta\sin^2\gamma(\cos^2\alpha + \cos^2\beta) \\
&\quad + \sin^2\xi\cos^2\eta\sin^2\alpha(\cos^2\beta + \cos^2\gamma)] \\
f_2 &= -[\cos^2\alpha\cos^2\beta\cos^2\gamma + \lambda^2(\sin^2\xi\sin^2\eta\cos^2\alpha\cos^2\beta\sin^2\gamma \\
&\quad + \sin^2\xi\cos^2\eta\sin^2\alpha\cos^2\beta\cos^2\gamma + \cos^2\xi\cos^2\alpha\sin^2\beta\cos^2\gamma)] \\
&\quad \times [(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) + \lambda^2(\sin^2\xi\sin^2\eta\sin^2\gamma \\
&\quad + \sin^2\xi\cos^2\eta\sin^2\alpha + \cos^2\xi\sin^2\beta)] \\
f_3 &= [\cos^2\alpha\cos^2\beta\cos^2\gamma + \lambda^2(\cos^2\xi\cos^2\alpha\sin^2\beta\cos^2\gamma + \sin^2\xi\sin^2\eta\cos^2\alpha\cos^2\beta\sin^2\gamma \\
&\quad + \sin^2\xi\cos^2\eta\sin^2\alpha\cos^2\beta\cos^2\gamma)]^2. \tag{2.11}
\end{aligned}$$

Although one can solve $\Lambda$ analytically in principle, it would be too lengthy to express them explicitly. When, however, $\alpha = \gamma$, the eigenvalues reduce to the simpler expression in the following:

$$\begin{aligned}
\Lambda_1 &= \cos^2\alpha\cos^2\beta + \lambda^2[\cos^2\xi\cos^2\alpha\sin^2\beta + \sin^2\xi\sin^2\alpha\cos^2\beta] \\
\Lambda_{\pm} &= \frac{\cos^2\alpha}{2}[(\cos^2\alpha + \cos^2\beta) + \lambda^2(\cos^2\xi\sin^2\beta + \sin^2\xi\sin^2\alpha) \pm \tilde{\Lambda}]
\end{aligned} \tag{2.12}$$

where

$$\tilde{\Lambda} = \sqrt{\{(\cos^2\alpha - \cos^2\beta) + \lambda^2(\cos^2\xi\sin^2\beta + \sin^2\xi\sin^2\alpha)\}^2 - 4\lambda^2\cos^2\xi\sin^2\beta(\cos^2\alpha - \cos^2\beta)}. \tag{2.13}$$

Another special case is $\xi = 0$, which gives

$$\Lambda_1 = \cos^2\alpha(\cos^2\beta + \lambda^2\sin^2\beta) \qquad \Lambda_2 = \cos^2\gamma(\cos^2\beta + \lambda^2\sin^2\beta) \qquad \Lambda_3 = \cos^2\alpha\cos^2\gamma. \tag{2.14}$$

These eigenvalues will be used to analyze the amplitude damping channels simulated by the single-qubit environment.

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## 3. Generalized amplitude damping

Amplitude damping is a description of energy dissipation effects due to loss of energy from a quantum system. The operator-sum representation for the amplitude damping channel defined in equation (1.1) can be generalized by

$$\rho \rightarrow \rho' = \sum_{i=0}^{3} E_i \rho E_i^{\dagger} \tag{3.1}$$

where the operation elements are

$$E_0 = \sqrt{\epsilon_0} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\gamma_0} \end{pmatrix}, \qquad E_1 = \sqrt{\epsilon_1} \begin{pmatrix} 0 & \sqrt{\gamma_1} \\ 0 & 0 \end{pmatrix}$$
$$E_2 = \sqrt{\epsilon_2} \begin{pmatrix} \sqrt{\gamma_2} & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_3 = \sqrt{\epsilon_3} \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma_3} & 0 \end{pmatrix} \tag{3.2}$$

with

$$\epsilon_0 + \gamma_2\epsilon_2 + \gamma_3\epsilon_3 = \gamma_0\epsilon_0 + \gamma_1\epsilon_1 + \epsilon_2 = 1. \tag{3.3}$$

The fact $\sum_i E_i^{\dagger} E_i = I$ implies that the quantum operation for the amplitude damping is a trace-preserving map. Since there are four operation elements, the GAD is realized when the environment is a two-qubit system. Therefore, a natural question arises: how much portion for the amplitude damping can be simulated when the environment is a single-qubit mixed state? This question is related to the volume issue, which will be discussed in the following section.

The amplitude damping defined in equation (3.1) can be described by the affine map

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{pmatrix} = M_{AD} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + \vec{C}_{AD} \tag{3.4}$$

where

$$M_{AD} = \begin{pmatrix} \epsilon_0\sqrt{\gamma_0} + \epsilon_2\sqrt{\gamma_2} & 0 & 0 \\ 0 & \epsilon_0\sqrt{\gamma_0} + \epsilon_2\sqrt{\gamma_2} & 0 \\ 0 & 0 & -1 + \epsilon_0(1+\gamma_0) + \epsilon_2(1+\gamma_2) \end{pmatrix}$$
$$\vec{C}_{AD} = \begin{pmatrix} 0 \\ 0 \\ \epsilon_0(1-\gamma_0) - \epsilon_2(1-\gamma_2) \end{pmatrix}. \tag{3.5}$$

Thus the generalized amplitude damping has following two important properties: (i) the transformation matrix $M_{AD}$ has two-fold degeneracy in the eigenvalues. (ii) the first two components of the translation vector $\vec{C}_{AD}$ are zero. As shown in equation (1.3) the standard amplitude damping has same properties. This is a reason why we define the GAD as equations (3.1) and (3.2).

The most general GAD channels simulated from the two-qubit environment can be represented by the three-dimensional volume $(X_2, Y_2, Z_2)$ defined as

$$\begin{aligned} X_2 &= \epsilon_0\sqrt{\gamma_0} + \epsilon_2\sqrt{\gamma_2} \\ Y_2 &= -1 + \epsilon_0(1+\gamma_0) + \epsilon_2(1+\gamma_2) \\ Z_2 &= \epsilon_0(1-\gamma_0) - \epsilon_2(1-\gamma_2). \end{aligned} \tag{3.6}$$
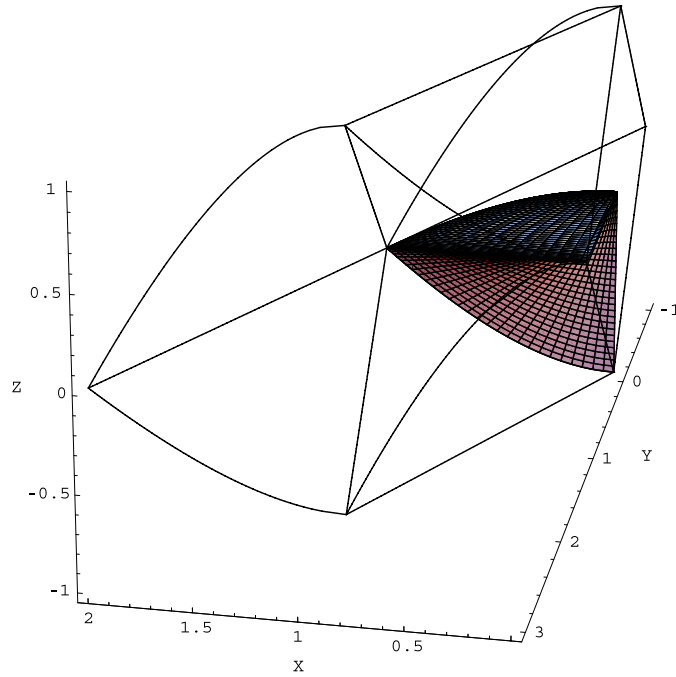
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**Figure 2.** Graphical representation of the volumes occupied by $(X_2, Y_2, Z_2)$ (transparent volume) and $(X_1, Y_1, Z_1)$ (opaque volume). As expected the opaque volume is embedded into the transparent volume. This fact indicates that the amplitude damping channel cannot be simulated completely by the one-qubit environment.

(This figure is in colour only in the electronic version)

This volume is plotted in figure 2 transparently to compare with the volume derived from the single-qubit mixed-state environment[4]. Compared to the depolarizing channel, where the tetrahedron volume is derived [16], the volume for the amplitude damping channel is very complicated. Since, furthermore, $(X_2, Y_2, Z_2)$ depends on the four parameters $\epsilon_0$, $\epsilon_2$, $\gamma_0$ and $\gamma_2$, it is highly difficult to compute the volume exactly. The numerical calculation gives the volume approximately 1.67. We will show in the following section that the volume derived from the single-qubit mixed-state environment is embedded in this volume.

## 4. Volume issue

In this section, we want to explore the amplitude damping when the environment is a single-qubit mixed-state. In order to simulate the amplitude damping, as shown in the previous section, the transformation matrix (2.8) should have the following two properties: (i) in the singular value decomposition $M = UDV$ where $U$ and $V$ are unitary matrices, the diagonal matrix $D$ should have double degeneracy; (ii) the first two components of the translation vector $U^{\dagger}\vec{C}$ should be zero.

In this paper, we consider the case of $\alpha = \gamma$, where the eigenvalues of $M^{\dagger}M$ are somewhat simple. In this case, equations (2.12) and (2.13) imply that the necessary condition for the diagonal matrix $D$ to have the double degeneracy is the removal of the square root in $\tilde{\Lambda}$. This

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[4] The three-dimensional animation of the volumes $(X_1, Y_1, Z_1)$ and $(X_2, Y_2, Z_2)$ can be seen in http://rose.kyungnam.ac.kr/paper.htm.

**Table 1.** The diagonal components of $D$ for each case. The double-degeneracy occurs for the cases of $\xi = 0$ and $\alpha = \beta$.

| Cases | Diagonal components |
|---|---|
| $\xi = 0$ | $D_{11} = D_{22} = \cos\alpha\sqrt{\cos^2\beta + \lambda^2\sin^2\beta}$ |
| | $D_{33} = \cos^2\alpha$ |
| $\xi = \frac{\pi}{2}$ | $D_{11} = \cos\alpha\sqrt{\cos^2\alpha + \lambda^2\sin^2\alpha}$ |
| | $D_{22} = \cos\beta\sqrt{\cos^2\alpha + \lambda^2\sin^2\alpha}$ |
| | $D_{33} = \cos\alpha\cos\beta$ |
| $\beta = 0$ | $D_{11} = \sqrt{\cos^2\alpha + \lambda^2\sin^2\xi\sin^2\alpha}$ |
| | $D_{22} = \cos\alpha\sqrt{\cos^2\alpha + \lambda^2\sin^2\xi\sin^2\alpha}$ |
| | $D_{33} = \cos\alpha$ |
| $\alpha = \beta$ | $D_{11} = D_{22} = \cos\alpha\sqrt{\cos^2\alpha + \lambda^2\sin^2\alpha}$ |
| | $D_{33} = \cos^2\alpha$ |

condition reduces to the following four distinct cases: (1) $\xi = 0$, (2) $\xi = \pi/2$, (3) $\beta = 0$, (4) $\alpha = \beta = \gamma$. The diagonal components $D_{11}$, $D_{22}$, and $D_{33}$ for the diagonal matrix $D$ for each case are summarized in table 1.

Table 1 indicates that the cases $\xi = \pi/2$ and $\beta = 0$ are excluded as candidates for the amplitude damping due to no degeneracy. The singular value decomposition for the remaining candidates are for $\xi = 0$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \cos\alpha\sqrt{\cos^2\beta + \lambda^2\sin^2\beta} & 0 & 0 \\ 0 & \cos\alpha\sqrt{\cos^2\beta + \lambda^2\sin^2\beta} & 0 \\ 0 & 0 & \cos^2\alpha \end{pmatrix} \quad (4.1)$$

$$V = \frac{1}{\sqrt{\cos^2\beta + \lambda^2\sin^2\beta}}\begin{pmatrix} \lambda\sin\beta & -\cos\beta & 0 \\ \cos\beta & \lambda\sin\beta & 0 \\ 0 & 0 & \sqrt{\cos^2\beta + \lambda^2\sin^2\beta} \end{pmatrix}$$

and for $\alpha = \beta$

$$U = \frac{1}{\sqrt{\phantom{x}}}\begin{pmatrix} \sin\eta\cos\alpha - \lambda\cos\xi\cos\eta\sin\alpha & \cos\xi\cos\eta\cos\alpha + \lambda\sin\eta\sin\alpha & \sin\xi\cos\eta\sqrt{\phantom{x}} \\ -(\cos\eta\cos\alpha + \lambda\cos\xi\sin\eta\sin\alpha) & \cos\xi\sin\eta\cos\alpha - \lambda\cos\eta\sin\alpha & \sin\xi\sin\eta\sqrt{\phantom{x}} \\ \lambda\sin\xi\sin\alpha & -\sin\xi\cos\alpha & \cos\xi\sqrt{\phantom{x}} \end{pmatrix}$$

$$D = \begin{pmatrix} \cos\alpha\sqrt{\phantom{x}} & 0 & 0 \\ 0 & \cos\alpha\sqrt{\phantom{x}} & 0 \\ 0 & 0 & \cos^2\alpha \end{pmatrix} \quad (4.2)$$

$$V = \begin{pmatrix} \sin\eta & -\cos\eta & 0 \\ \cos\xi\cos\eta & \cos\xi\sin\eta & -\sin\xi \\ \sin\xi\cos\eta & \sin\xi\sin\eta & \cos\xi \end{pmatrix}$$

respectively, where $\sqrt{\phantom{x}} = \sqrt{\cos^2\alpha + \lambda^2\sin^2\alpha}$. Computing the translation vector $U^\dagger\vec{C}$, one can show that the amplitude damping derived from the single-qubit mixed-state environment is

represented by the three-dimensional volume $(X_1, Y_1, Z_1)$ defined as

$$X_1 \equiv D_{11} = \cos\alpha\sqrt{\cos^2\beta + \lambda^2\sin^2\beta}$$
$$Y_1 \equiv D_{33} = \cos^2\alpha \qquad\qquad\qquad (4.3)$$
$$Z_1 \equiv (U^\dagger \vec{C})_3 = -\lambda\sin^2\alpha.$$

The volume generated by $(X_1, Y_1, Z_1)$ is plotted in figure 2 opaquely (see footnote 4). As expected this volume is embedded in the lucid volume generated by $(X_2, Y_2, Z_2)$. This means that the amplitude damping channel cannot be completely simulated by the one-qubit environment although it is in the arbitrary mixed-state as depolarizing channel. The volume for $(X_1, Y_1, Z_1)$ can be computed analytically, which is $2/15$. Thus the volume ratio, i.e. opaque volume divided by transparent volume, is approximately 0.08. This is much smaller than $3/8$, which is the volume ratio for the depolarizing channel.

## 5. Conclusion

We have studied the GAD channels simulated by the one-qubit mixed-state environment when the principal system is initially in the single-qubit pure state. Examing the affine map for the GAD channel simulated by the two-qubit environment, we have found that $\xi = 0$ with $\alpha = \gamma$ and $\alpha = \beta = \gamma$ are the GAD channel simulated by the one-qubit mixed-state environment. Representing the affine map as a three-dimensional volume, we have plotted the volume opaquely in figure 2. As expected, this volume is embedded in the total volume generated by the two-qubit environment. It turns out that the volume ratio is much smaller than $3/8$, which is the volume ratio for the depolarizing channel.

It seems to be interesting to explore the various different damping channels in this way. For example, let us consider the phase damping whose quantum operation is defined as $\varepsilon(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$, where operation elements are

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \qquad\qquad (5.1)$$

and $\lambda$ is a quantity related to a relaxation time. The affine map for the phase damping is thus $(r_1, r_2, r_3) \rightarrow (r_1\sqrt{1-\lambda}, r_2\sqrt{1-\lambda}, r_3)$. Therefore, the effect of the phase damping is to shrink the Bloch sphere into ellipsoid. To explore the effect of the one-qubit mixed-state environment in the phase damping process firstly we should generalize it by introducing four operation elements considering the general features of the standard phase damping. Next we should find same channels when the environment is a single-qubit mixed states by making use of equation (2.8). It is unclear at least for us how to construct the generalized phase damping.

Another direction we would like to explore is to compute the entanglement measure when the environment is involved. Recently, the Groverian measure for mixed states was introduced in [18]. Although it was shown in [18] that the Groverian measure for mixed states is entanglement monotone, the explicit computation of it for given mixed states is highly nontrivial mainly due to the maximization over purification while the analytic computation for the pure states is sometimes possible [19]. Since environment in general makes the state of quantum system mixed state, it seems to be highly interesting to explore the role of entanglement in the damping process.

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