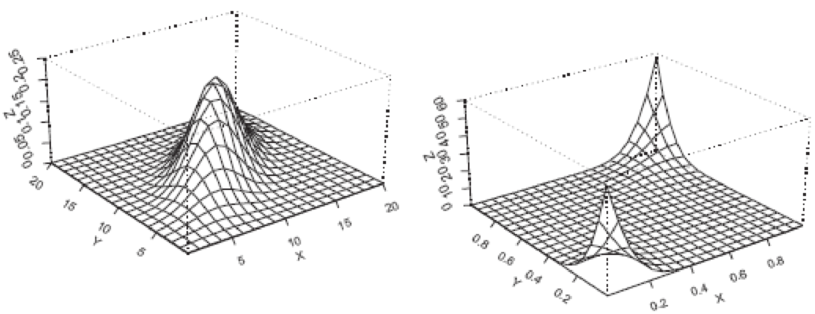
An Introduction to Copulas

: Which Copula is the right one?

Dependence Concepts
Copula Families
Elliptical Copula
Archimedean Copula
Kendall's tau
Spearman Rho
Dependency Structure



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Expect the unexpected

For Credit Risk

Copula의 유용성

□ Risk Factor 간에 의존성(Dependency)을 측정

□ 선형 상관관계는 선형관계의 의존성을 나타내지만 실제 금융시장간의 의존성은 비선형관계(Non-linear dependency)를 인식

- **Elliptical distributions** - linear dependence structure - correlation coefficient meaningful
- **Non-elliptical distributions** - alternative measures of dependence needed ⇒ Copulas

□ 다변량 분포들은 Copula를 이용하면 쉽게 계산가능

역사

□ **1940's : Hoeffding studies properties of multivariate distributions**
□ **1959 : The word copula appears for the first time (Sklar)**
□ **1999 : Introduced to financial applications (Embrechts, McNeil, Straumann)**
□ **2006 : Several insurance companies, banks and other financial institutions apply copula as a Risk management, Credit Derivatives, CDO, Options etc**

Copula 함수의 이해

- Copula 함수는 복잡한 종속성 구조(**dependence structure**)를 고려하면서 다변량 누적분포함수를 추정하는데 유용한 도구 (is a function that links univariate marginals to their full multivariate distribution)
- Copula 함수는 다변량 분포함수와 단변량 한계분포함수(margins)를 연결시키는 함수를 의미하며 데이터 사이의 종속성 구조와 개별자료의 분포를 분리하여 모형화함으로써 추정과 시뮬레이션을 용이

□ **CDF**

$$F(x_1, x_2) = P[X_1 \le x_1, X_2 \le x_2] \qquad F(x_2) = P[X_2 \le x_2] \qquad F(x_1) = P[X_1 \le x_1]$$

□ **Copula fun (Sklar's theorem)**

$$C(u_1, u_2) = F(x_1, x_2) \qquad C^2(F_1(x_1), F_2(x_2)) = F(x_1, x_2)$$

□ **Property of copula**

$$C(u_1, u_2) \qquad 0 \le u_1, u_2 \le 1$$

$$0 \le C(u_1, u_2) \le 1$$

$$x_i = F_i(u_i)^{-1}$$

$$F_i(x_i) = u_i$$

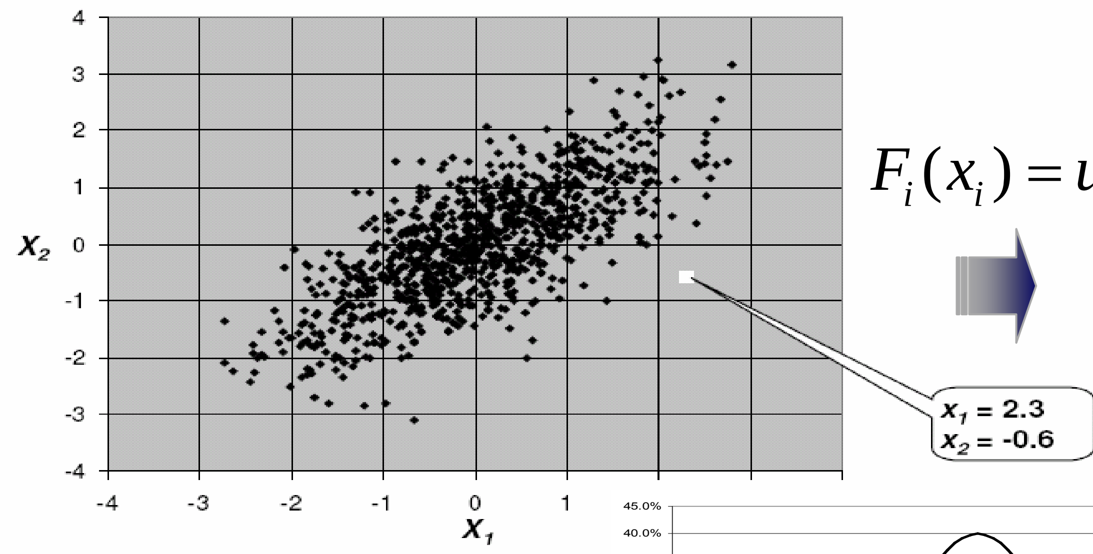
$$C(u_1, 1) = u_1 \qquad C(1, u_2) = u_2$$

$$C(u_1, 1) = F(x_1, x_2) \quad P[X_1 \le x_1] = u_1 \quad P[X_2 \le x_2] = 1 \quad x_2 = \infty$$

$$C(u_1, 1) = F(x_1, x_2) = P[X_1 \le x_1, X_2 \le \infty] = P[X_1 \le x_1] = u_1$$

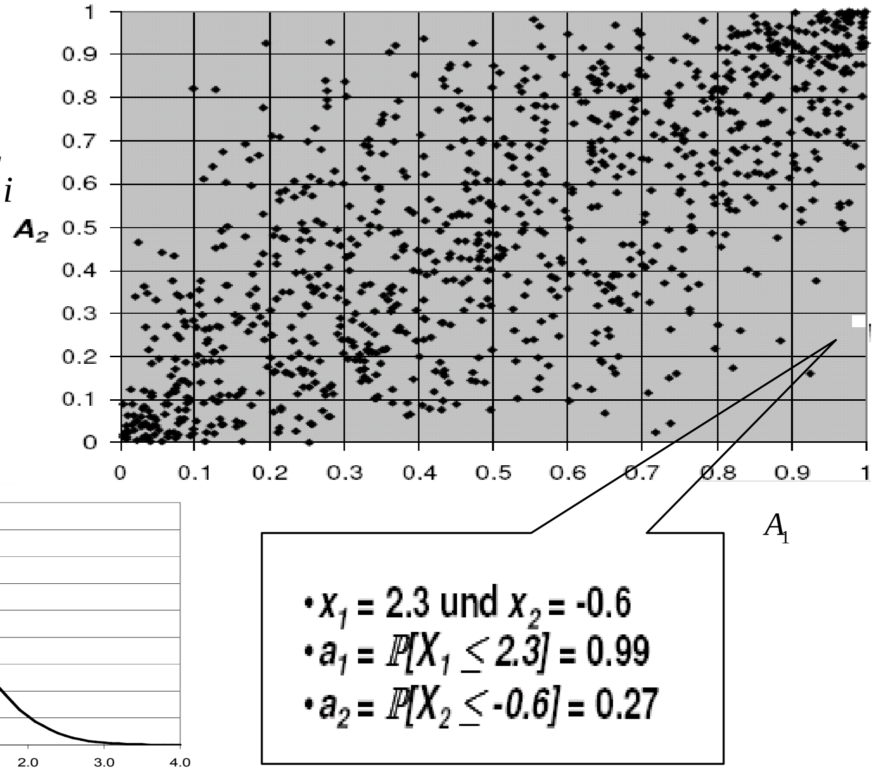
$$C(u_1, 0) = [0, u_1] \times [0, 0] = 0$$

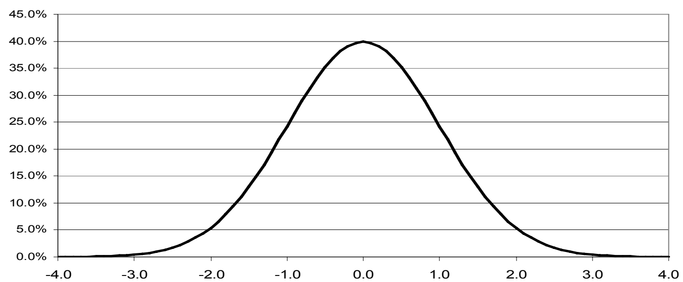
- Standard Normal function : N(0,1)



$$F_i(x_i) = u_i$$

- Copula Function





$$C(0.1, 0.2) = F(-1.28, -0.84) = F(x_1, x_2) = P[X_1 \le -1.28, X_2 \le -0.84] = 6.90\%$$

$$= [0, 0.1] \times [0, 0.2] / 1000$$ □ 각 길이의 **product**

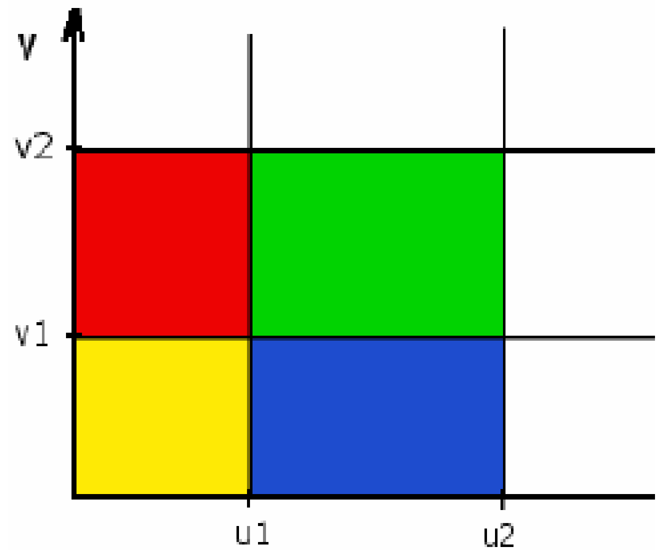
Where $F_1(x_1) = P[X_1 \le x_1] = 0.1$ $\quad X_1 = -1.28$

2 dimension copula

$$C:[0,1]\times[0,1]\to[0,1] \qquad u_1,u_2,v_1,v_2\in[0,1]$$

$$u_1\le u_2 \text{ and } v_1\le v_2$$

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\ge 0$$



For all $(a_1,\ldots,a_n),(b_1,\ldots,b_n)\in[0,1]^n$ with $a_i\le b_i$ we have:

$$\sum_{i_1=1}^{2}\cdots\sum_{i_n=1}^{2}(-1)^{i_1+\cdots+i_n}C(x_{1i_1},\ldots,x_{ni_n})\ge 0\ ,$$

where $x_{j1}=a_j$ and $x_{j2}=b_j$ for all $j\in\{1,\ldots,n\}$.

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}(-1)^{i_1+i_2}C(x_{1i_1},x_{2i_2})\ge 0$$

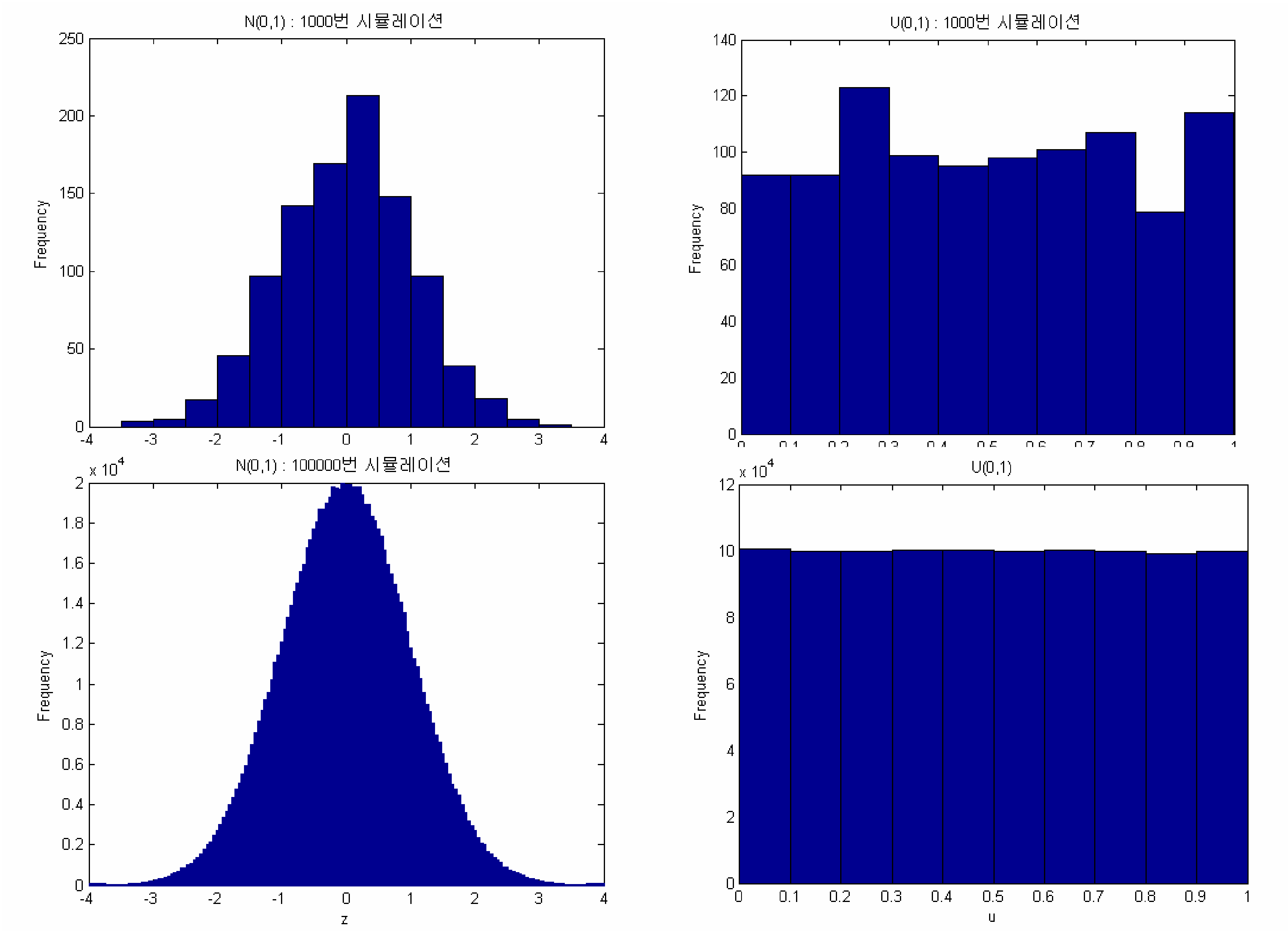
$$\Leftrightarrow \sum_{i_1=1}^{2}[(-1)^{i_1+1}C(x_{1i_1},x_{21})+(-1)^{i_1+2}C(x_{1i_1},x_{22})]\ge 0$$

$$\Leftrightarrow (-1)^2C(x_{11},x_{21})+(-1)^3C(x_{11},x_{22})+(-1)^3C(x_{12},x_{21})+(-1)^4C(x_{12},x_{22})\ge 0$$

$$\Leftrightarrow C(x_{11},x_{21})-C(x_{11},x_{22})-C(x_{12},x_{21})+C(x_{12},x_{22})\ge 0$$

with $x_{j1}=a_j$ and $x_{j2}=b_j$ for all $j\in\{1,2\}$ it follows:

$$C(a_1,a_2)-C(a_1,b_2)-C(b_1,a_2)+C(b_1,b_2)\ge 0 \qquad (2.3)$$



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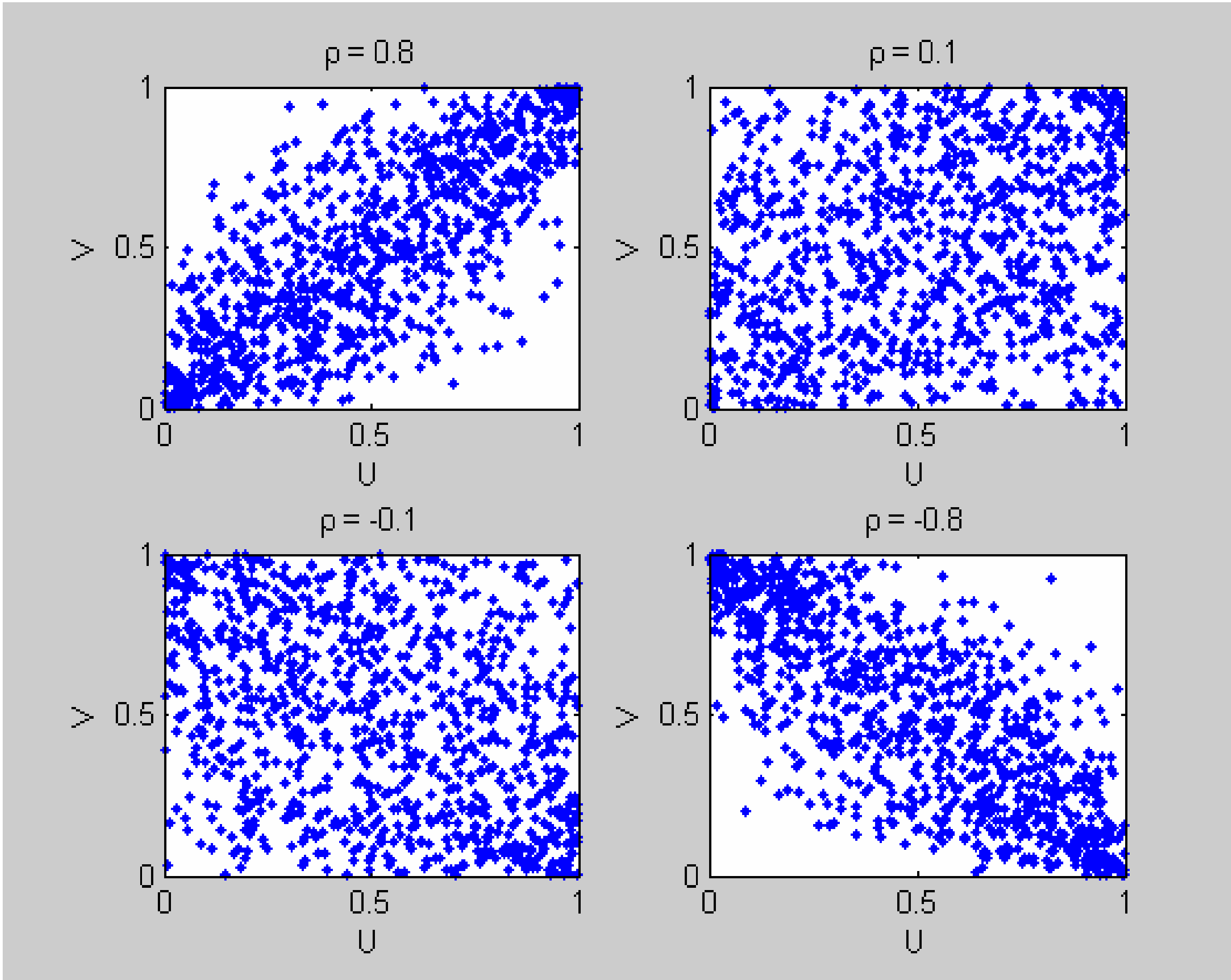
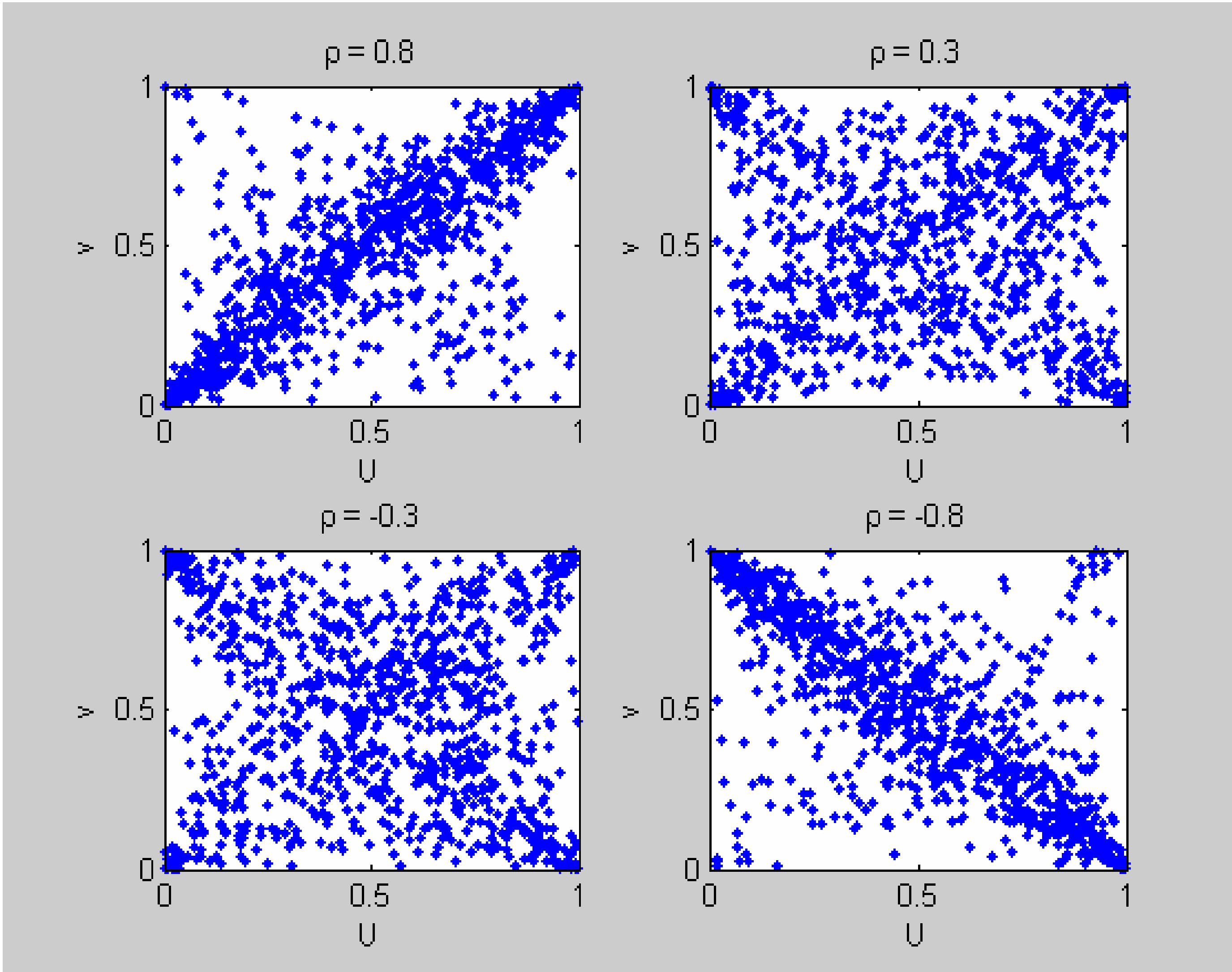
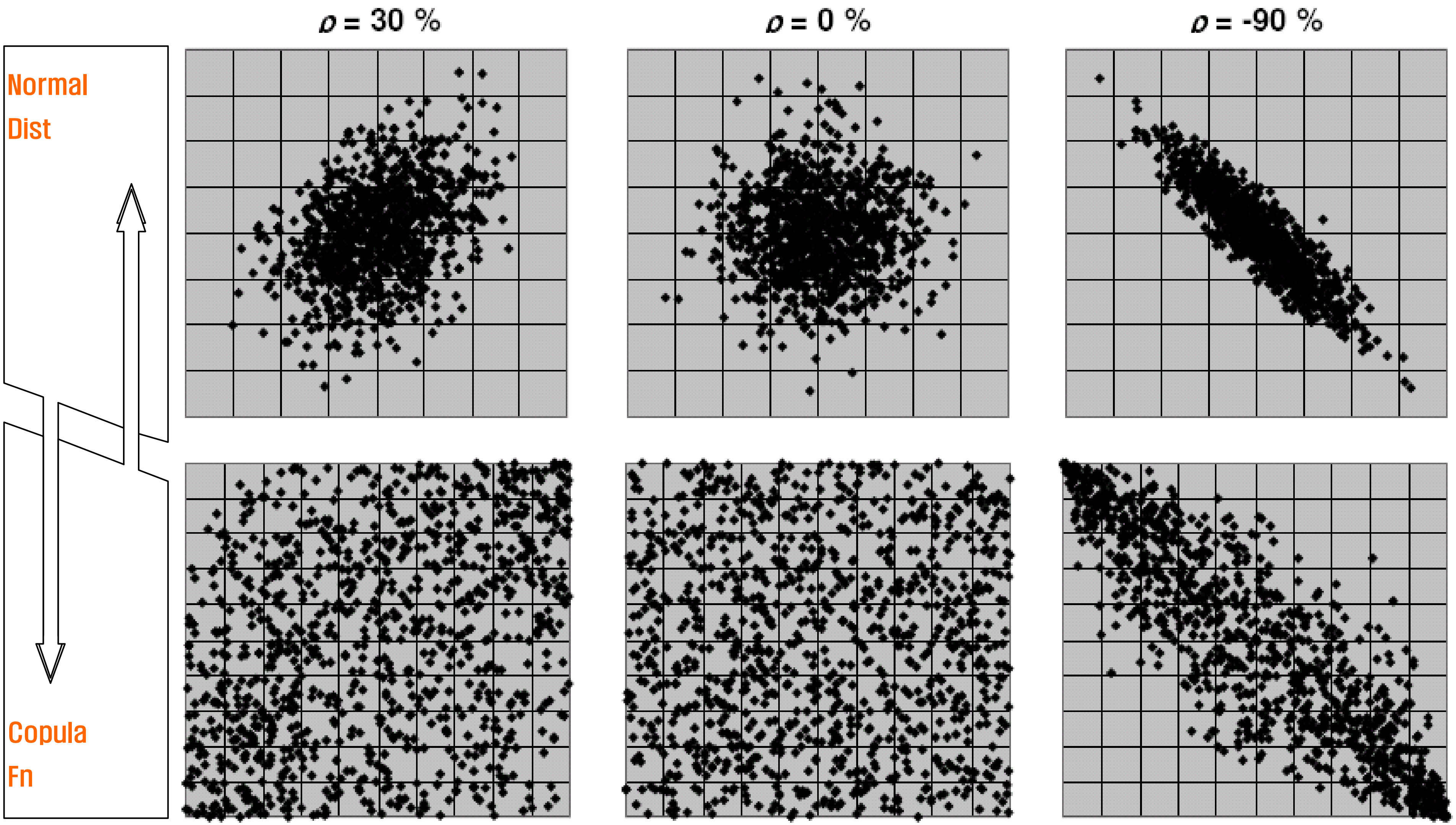

ρ = 0.8
ρ = 0.1
ρ = -0.1
ρ = -0.8
V
U
0
0.5
1


ρ = 0.8
ρ = 0.3
ρ = -0.3
ρ = -0.8
v
u
0
0.5
1

Copula 함수의 이해

□ Copula function vs Cumulative Distribution :Normal case



Multivariate Copula Function

$$C(F_1(x_1), F_2(x_2), \cdots, F_m(x_m), \rho) = \Pr[U_1 \le F_1(x_1), U_2 \le F_2(x_2), \cdots, U_m \le F_m(x_m)]$$
$$= \Pr[F_1^{-1}(U_1) \le x_1, F_2^{-1}(U_2) \le x_2, \cdots, F_m^{-1}(U_m) \le x_m]$$
$$= \Pr[X_1 \le x_1, X_2 \le x_2, \cdots, X_m \le x_m]$$
$$= F(x_1, x_2, \cdots, x_m)$$

The marginal ditribution of X_i

$$C(F_1(+\infty), F_2(+\infty), \cdots, F_2(x_i), \cdots F_m(+\infty), \rho) = \Pr[F_1(+\infty), F_2(+\infty), \cdots, F_i(x_i) \cdots F_m(+\infty), \rho]$$
$$= \Pr[X_1 \le +\infty, X_2 \le +\infty, \cdots, X_i \le x_i, \cdots X_m \le +\infty]$$
$$= \Pr[X_i \le x_i]$$
$$= F(x_i)$$

$$\Rightarrow \quad F(x_1, x_2, \cdots, x_m) = C(F_1(x_1), F_2(x_2), \cdots, F_m(x_m), \rho)$$

$$\Rightarrow \quad C(u_1, u_2, \cdots, u_m) = \Phi_m(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \cdots, \Phi^{-1}(u_m); \rho)$$

포트폴리오의 함수 : Correlation 반영

Copula 함수 : 다변량 균등분포(Uniform Distribution)확률 변수의 결합 누적 확률 분포

$$C(u_1, u_2, \ldots\ldots u_n) = \Pr[X_1 \le x_1, X_2 \le x_2 \ldots., X_n \le x_n]$$

u_i : 균등분포인 확률변수

$$C(u_1, u_2, \ldots\ldots u_n)$$

$$u_1, u_2, \ldots\ldots u_n \in [0,1]^n$$

$$F_1(x_1), F_2(x_2) \ldots\ldots\ldots\ldots F_n(x_n)$$

$$F_i(x_i) = u_i$$

$$x_i = F_i(u_i)^{-1}$$

$$F(x_1, x_2, \ldots\ldots x_n) = C[F_1(x_1), F_2(x_2) \ldots\ldots\ldots\ldots F_n(x_n)]$$

다변량 누적 확률 분포

Sklar Representation 정리

$$C(x_1, x_2, \ldots\ldots x_n) = F[F_1^{-1}(x_1), F_2^{-1}(x_2) \ldots\ldots\ldots\ldots F_n^{-1}(x_n)]$$

$$C(\mathbf{u}) = P(U_1 \le u_1, U_2 \le u_2, \ldots, U_d \le u_d) = \int_0^{u_1} \cdots \int_0^{u_d} c(\mathbf{u}) \mathrm{d}\mathbf{u},$$

Copula 함수의 이해

Definition (Definition of copula function)

N-차원 Copula $C:[0,1]^n \to [0,1]$는 다음과 같은 특성을 만족하는 함수

(1) C 는 양의 확률(grounded)이고 n-증가 함수

(2) C 는 $u \in [0,1]$에 대하여 $C_i(u) = C(1,\ldots,1,u,1,\ldots,1) = u$

(3) C 는 n- 증가함수(increasing function)

- $f(x_1,\ldots,x_n)$는 다음과 같은 **2** 부분으로 분류 → **(1)**과 **(2)**의 결정하고 해당 모형에 모수를 결정

(1) Correlation structure

: **find the co-movement between variables may be viewed as a part of dependence structure**

(2) the product of marginal probability density functions

: **fit the individual marginal distribution and estimate their parameters (which can be achieved using regular statistical methods**

* Sklar의 정리는 copula 함수를 이용하여 종속성을 모형화하는 기본 아이디어를 제공
→ 어떤 다변량 분포함수에 대하여 단변량 한계분포와 종속성구조를 분리

Theorem (Sklar's Theorem)

$$F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$$

$$f(x_1,\ldots,x_n) = \frac{\partial F(x_1,\ldots,x_n)}{\partial x_1 \ldots \partial x_n} = \frac{\partial C(F_1(x_1),\ldots,F_n(x_n))}{\partial x_1 \ldots \partial x_n}$$

$$= \frac{\partial C(u_1,\ldots,u_n)}{\partial u_1 \ldots \partial u_n} \times \prod_i \frac{\partial F_i(x_i)}{\partial x_i}$$

$$= c(u_1,\ldots,u_n) \times \prod_i f_i(x_i) = c(\tilde{u}) \times \prod_i f_i(x_i)$$

$$c(\tilde{u}) = \frac{f(x_1,\ldots,x_n)}{\prod_i f_i(x_i)}$$

$f(x_1,\ldots,x_n)$ 는 $F(\cdot)$ 의 함수(probability density function)

$u_i = F_i(x_i) \quad i = 1,\ldots, \quad n$

$\tilde{u} = (u_1,\ldots,u_2)$

$c(\tilde{u})$ 는 Copula의 density function

** 모수 추정의 기초

Copula function property (bi-variate)

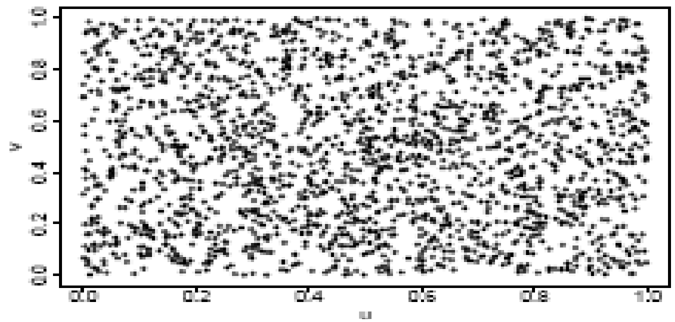
- $C(u, v)$ is between [0,1] and u, v are uniformly distributed [0,1]
- $C(0, v) = 0$
- $C(u, 1) = u$;
- $C(u, v)$ is a 2-increasing function

Special copula functions

- Minimum copula : $C(u, v) = \max(u+v-1, 0)$
- Maximum copula : $C(u, v) = \min(u, v)$
- Gaussian copula : $C(u, v) = \Phi_\rho(\Phi^{-1}(v), \Phi^{-1}(u))$
- Student t-copula : $C(u, v) = t_{\rho,s}(t_s^{-1}(u), t_s^{-1}(v))$
- **And more …**

Example 1: Independence copula

□ **If $U \sim U(0, 1)$ and $V \sim U(0, 1)$ are independent, then**
□ **$C(u, v) = uv = \; = P(U \le u)P(V \le v) = P(U \le u, V \le v) = H(u, v)$,**
□ **where $H(u, v)$ is the distribution function of (U, V). C is called the independence copula.**



Example 2: Gaussian copula

$$C_R^{GA}(u,v) = \int_{-\infty}^{\phi^{-1}(u)} \int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \exp\{-\frac{x^2 - 2R_{12}xy + y^2}{2(1-R_{12}^2)}\} dxdy$$

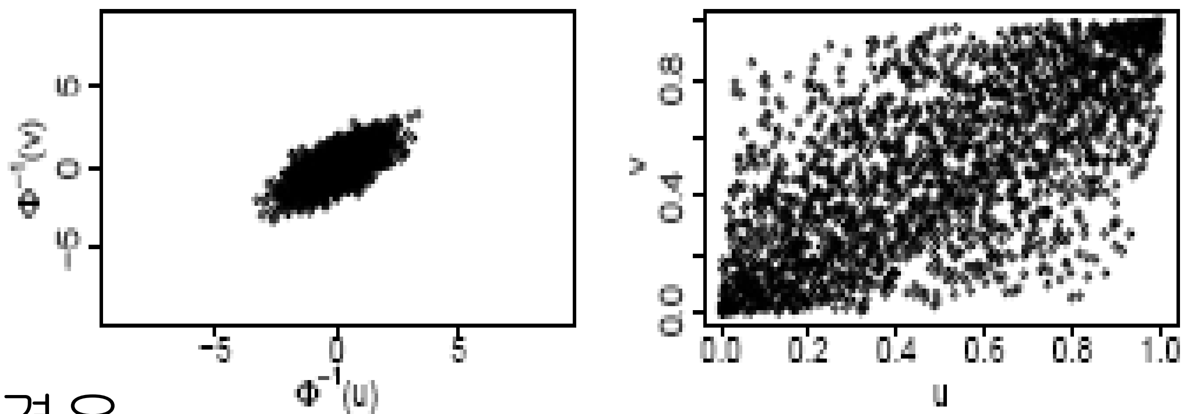
Example 3: Student's copula

$$C_{v,R}^{t}(u,v) = \int_{-\infty}^{t_v^{-1}(u)} \int_{-\infty}^{t_v^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \{1 + \frac{x^2 - 2R_{12}xy + y^2}{v(1-R_{12}^2)}\}^{-(v+2)/2} dxdy$$

□ ν is the degrees of freedom and **R** is **the linear correlation coefficient.**

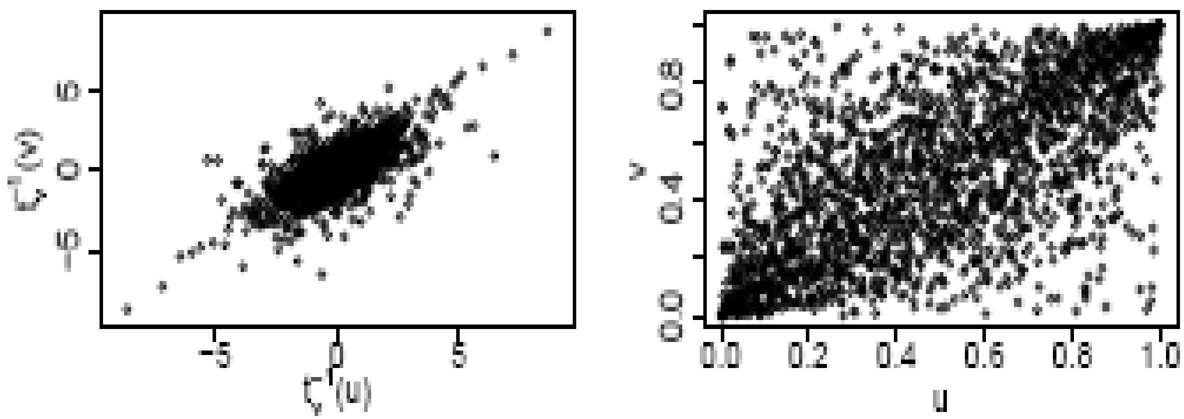
Gaussian copula $C_{\rho}^{Ga}(X) = \Phi_v^m(\Phi_v^{-1}(x_1), \Phi_v^{-1}(x_2), \cdots, \Phi_v^{-1}(x_m))$

Student's copula $C_{\rho}^{T}(X) = t_v^m(t_v^{-1}(x_1), t_v^{-1}(x_2), \cdots, t_v^{-1}(x_m))$



rho= 0.7

m=2인 경우



rho= 0.7, v = 4.

Archimedian Copula

□ Archimedean copula는 분석이 쉬울 뿐만 아니라(대부분의 Archimedean copula는 closed form expression을 가진다) 여러 가지 다른 종속성 구조를 제공

$\varphi : [0,1] \rightarrow [0,\infty]$

- φ는 연속함수이다.
- 모든 $u \in [0,1]$에 대하여, $\varphi'(u) < 0$.
- $\varphi(1) = 0$.

$\varphi^{-1} : [0,\infty] \rightarrow [0,1]$

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & for \quad 0 \le t \le \varphi(0) \\ 0 & for \quad \varphi(0) \le t \le \infty \end{cases}$$

□ φ 가 볼록성을 가지면 copula 함수는 다음과 같이 정의됨

함수 $C : [0,1]^2 \rightarrow [0,1]$:

$$C(u,v) = \varphi^{[-1]}[\varphi(u) + \varphi(v)]$$

Elliptical Copula

- Implied by well-known multivariate df's, derived through Sklar's theorem
- Extends the multivariate normal $\mathcal{N}_d(\boldsymbol{\mu}, \Sigma)$.
- Extend to arbitrary dimensions and are rich in parameters. A d-dim elliptical copula has at least $d(d-1)/2$ parameters
- Easy to simulate
- Drawback: Do not have closed form expressions and are restricted to have radial symmetry

Examples: Gaussian copula, Student's t copula

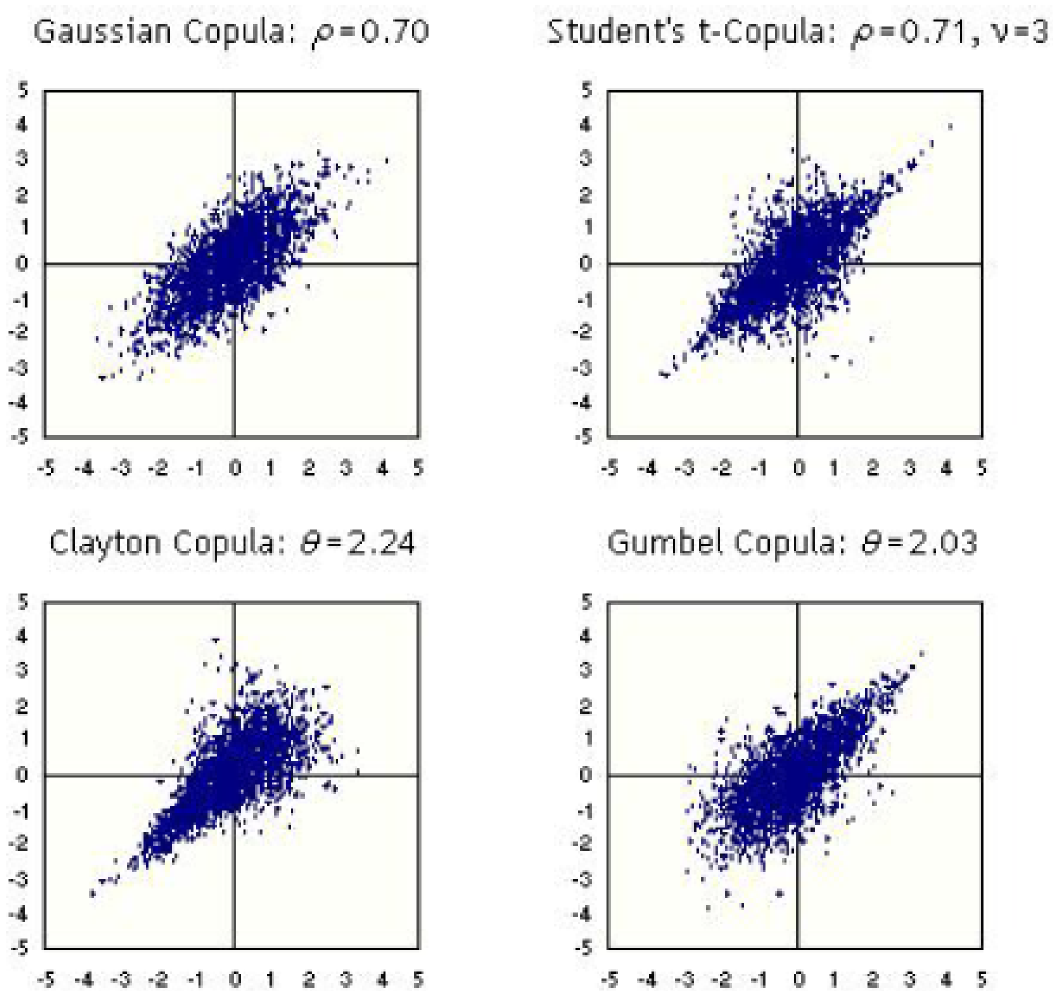
Archimedean Copula

The function φ is called the generator of the copula.

- Allow for a great variety of dependence structures
- Closed form expressions
- **Not** derived from mv df's using Sklar's theorem
- Drawback: Higher dimensional extensions difficult

Examples: Clayton copula, Gumbel copula

□ Dependency Structure



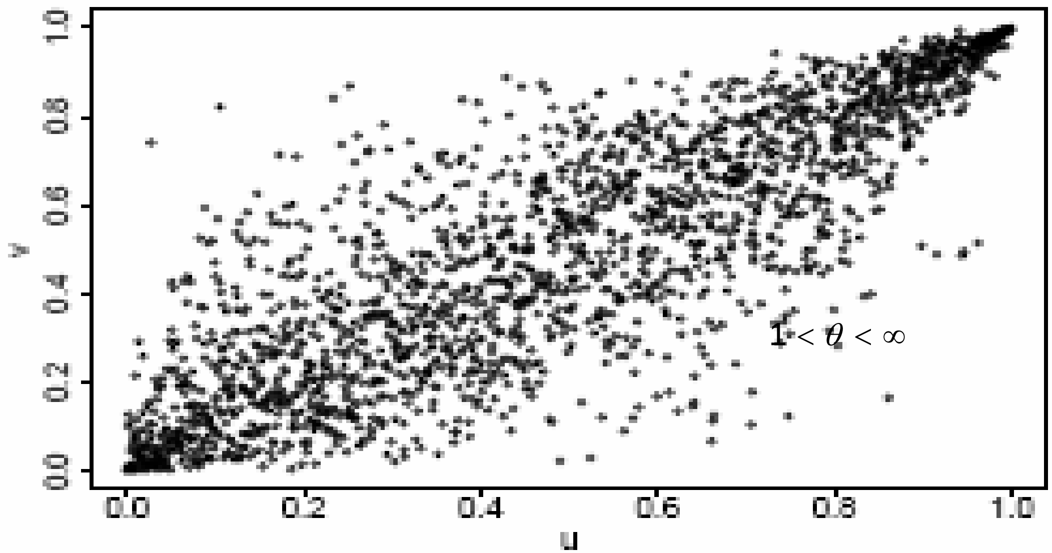
- ❑ Gaussian or t-Copula → Dependent Symetary 상승과 하락이 대칭
- ❑ Clayton Copula → Low tail dependence (하락 dependency)
- ❑ Gumbel Copula → Upper tail dependence(상승 dependency)

Example 4: Gumbel copula

$$C_\theta^{Gu}(u,v) = \exp\{-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}\}$$

$$\varphi(t) = (-\ln t)^\theta$$

$$1 \le \theta < \infty$$



$$C(u_1, \ldots, u_n) = \exp\left[-\left(\sum_{i=1}^{n}(-\ln u_i)^\alpha\right)^{1/\alpha}\right]$$

$$\theta = 3$$

$$\theta \to 1$$

Independence

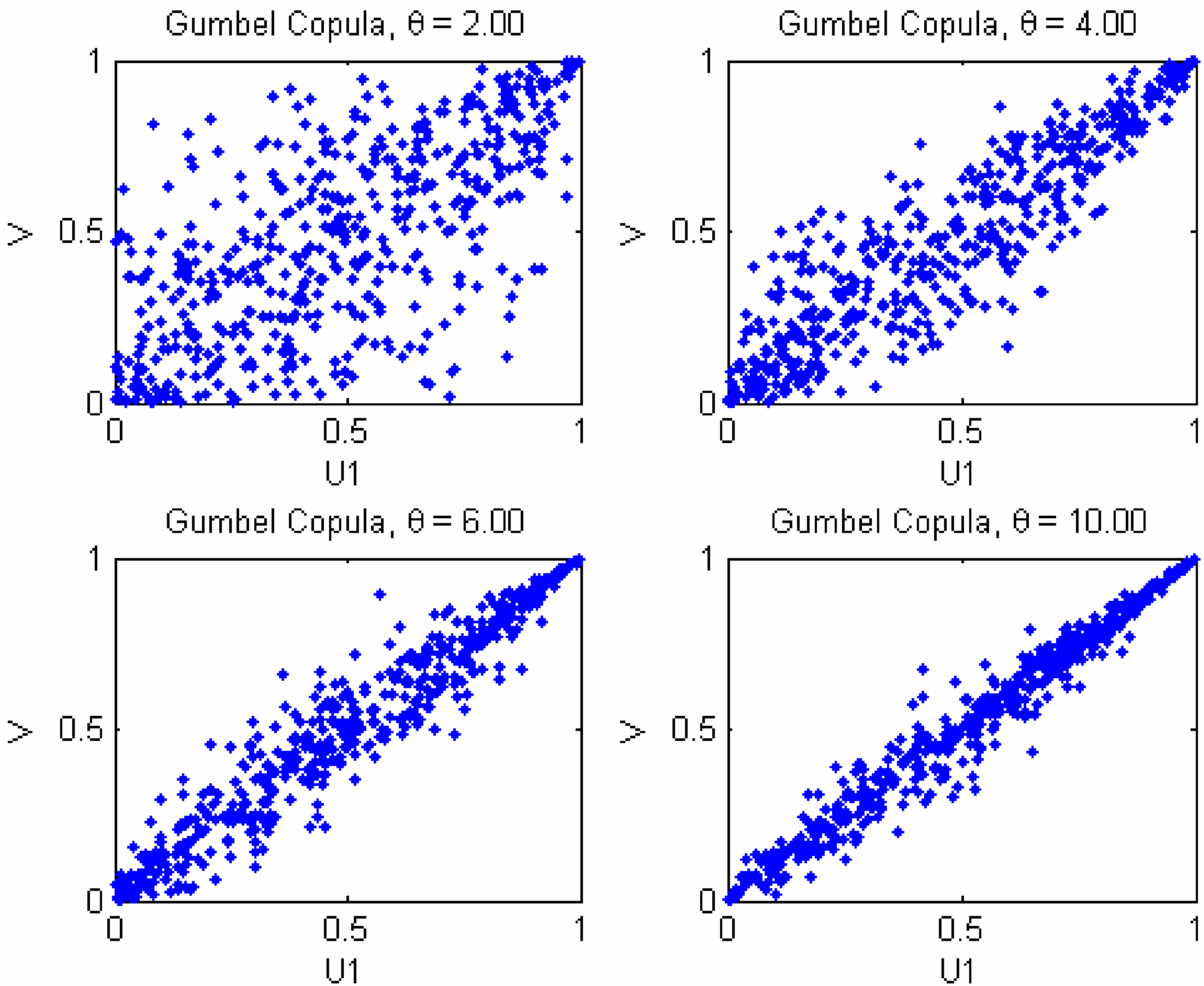
$$C_{\theta=1}^{Gu}(u,v) = \exp\{-[(-\ln u)^1 + (-\ln v)^1]^{1/1}\}$$
$$= \exp\{\ln(u) + \ln(v)\} = uv = \Pi$$

$$\theta \to \infty$$

$$C_{\theta=\infty}^{Gu}(u,v) = \min(u,v)$$

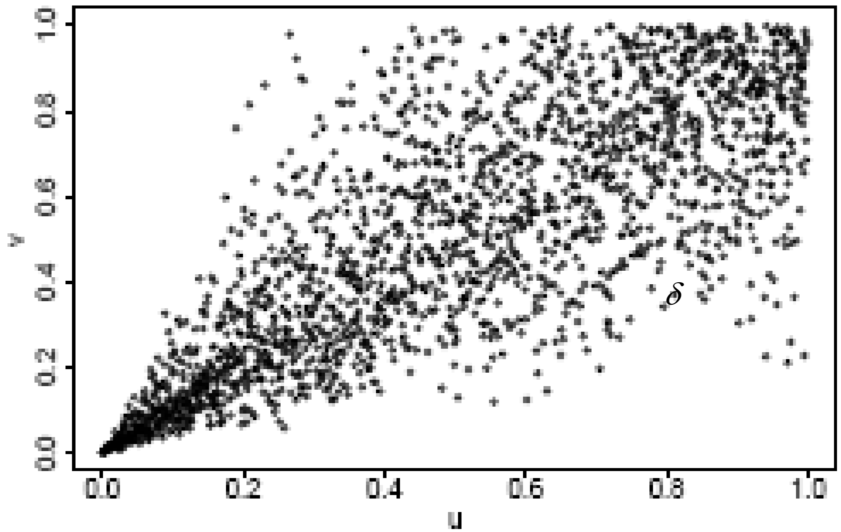
Perfect Dependence

Gumbel Copula



Example 5: Clayton copula (explicit)

$$C_{\theta}^{Clayton}(u,v) = (u^{\theta} + v^{\theta} - 1)^{-1/\theta} \qquad \varphi^{-1}(t) = \frac{t^{-\theta} - 1}{\theta} \qquad \theta > 0$$



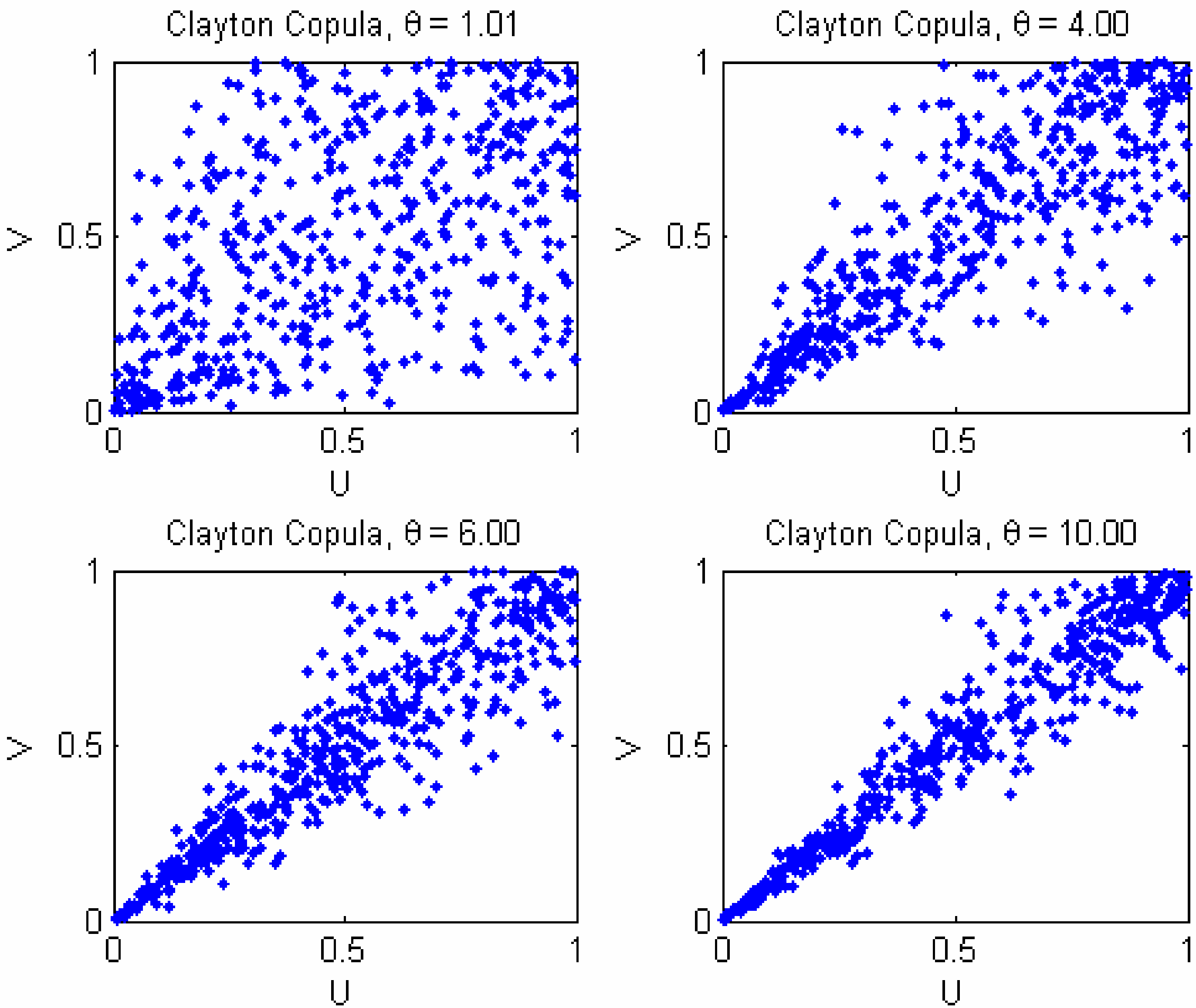
Bivariate Clayton copula (δ = 3).

$$C(u_1, \ldots, u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - 1\right]^{-1/\alpha}$$

$\theta \to 0$ **Independence**

$\theta \to \infty$ **Perfect Dependence**

Clayton Copula

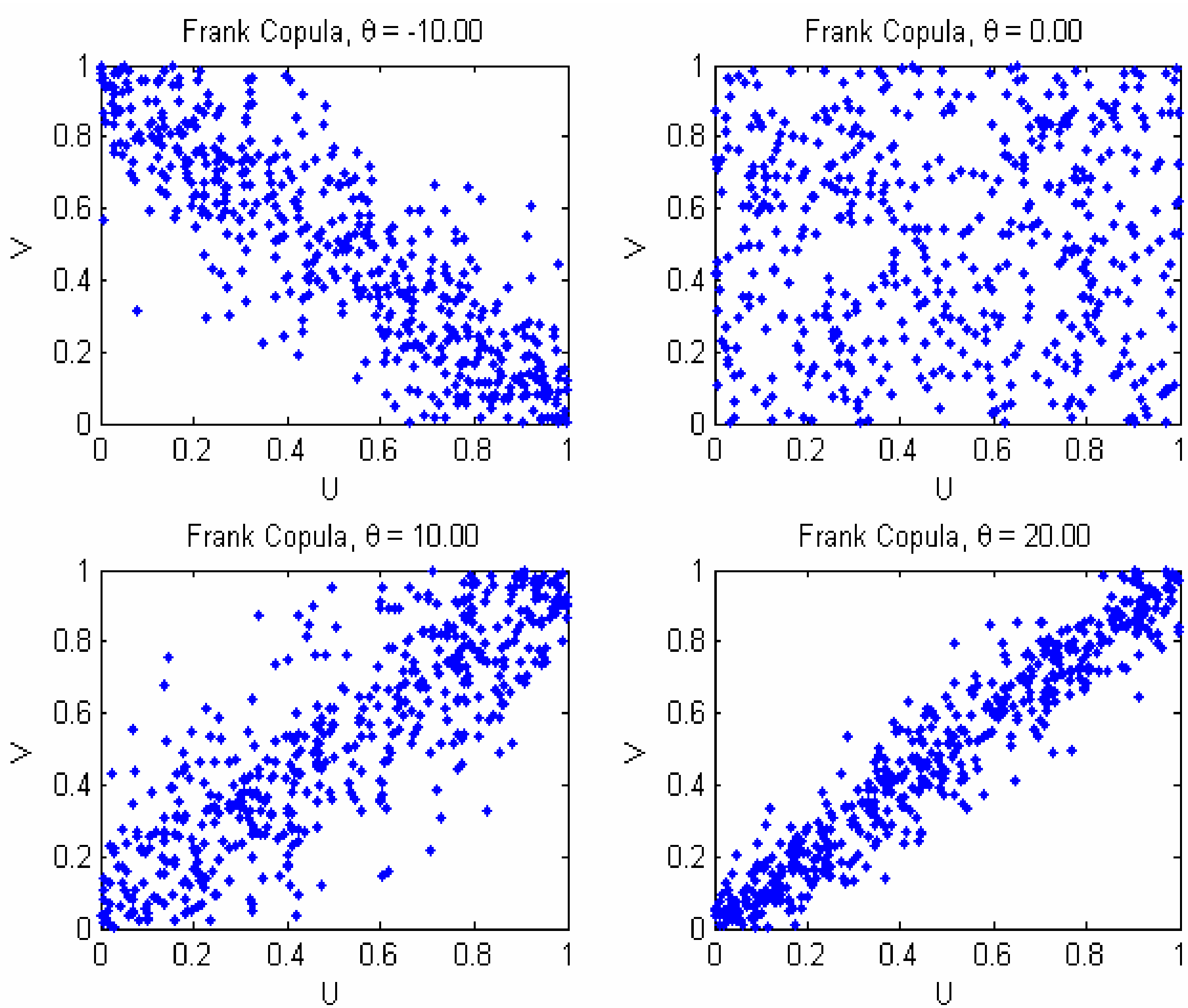


Example 5: Frank copula (explicit)

$$C_{\theta}^{Frank}(u,v) = -\frac{1}{\theta}\ln\left[1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{(e^{-\theta}-1)}\right]$$

$$\varphi^{-1}(t) = -\ln\frac{e^{\theta t}-1}{e^{\theta}-1}$$

$$-\infty < \theta < \infty$$



Dependency Structure

□ **Upper tail dependence**

$$\lambda_{upper} = \lim_{u\to 1} \Pr(Y \geq F_Y^{-1}(u) \,|\, X \geq F_X^{-1}(u))$$

$$= \frac{1-\Pr(Y \leq F_X^{-1}(u)) - \Pr(Y \leq F_Y^{-1}(u)) + \Pr(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u))}{1-\Pr(Y \leq F_X^{-1}(u))}$$

$$= \lim_{u\to 1} \frac{1-2u+C(u,u)}{1-u}$$

$$\Pr(Y \leq F_X^{-1}(u)) = \Pr(Y \leq F_Y^{-1}(u)) = u$$

$$\Pr(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u) = C(u,u)$$

□ **Lower tail dependence**

$$\lambda_{Lower} = \lim_{u\to 0} \Pr(Y \leq F_Y^{-1}(u) \,|\, X \leq F_X^{-1}(u))$$

$$= \frac{\Pr(X \leq F_X^{-1}(u), Y \leq F_Y^{-1}(u))}{\Pr(Y \leq F_X^{-1}(u))}$$

$$= \lim_{u\to 0} \frac{C(u,u)}{u}$$

□ **The tail area dependency measure depends on the copula and not on the marginal distribution**

Dependency Structure

- Gaussian Copula : the coefficients of lower tail and upper tail dependence

$$\lambda_U = \lambda_L = 2\lim_{x\to\infty}\Phi(x\sqrt{1-\rho}/\sqrt{1+\rho}) = 0$$

- T-copula : the coefficients of lower and upper tail dependence

$$\lambda_U = \lambda_L = 2t_{\nu+1}(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho})$$

- Clayton copula

$$\lambda_U = 0 \qquad \lambda_L = 2^{-1/\theta}$$

- Gumbel copula

$$\lambda_U = 2 - 2^{1/\theta} \qquad \lambda_L = 0$$

- Frank copula

$$\lambda_U = 0 \qquad \lambda_L = 0$$

Proof

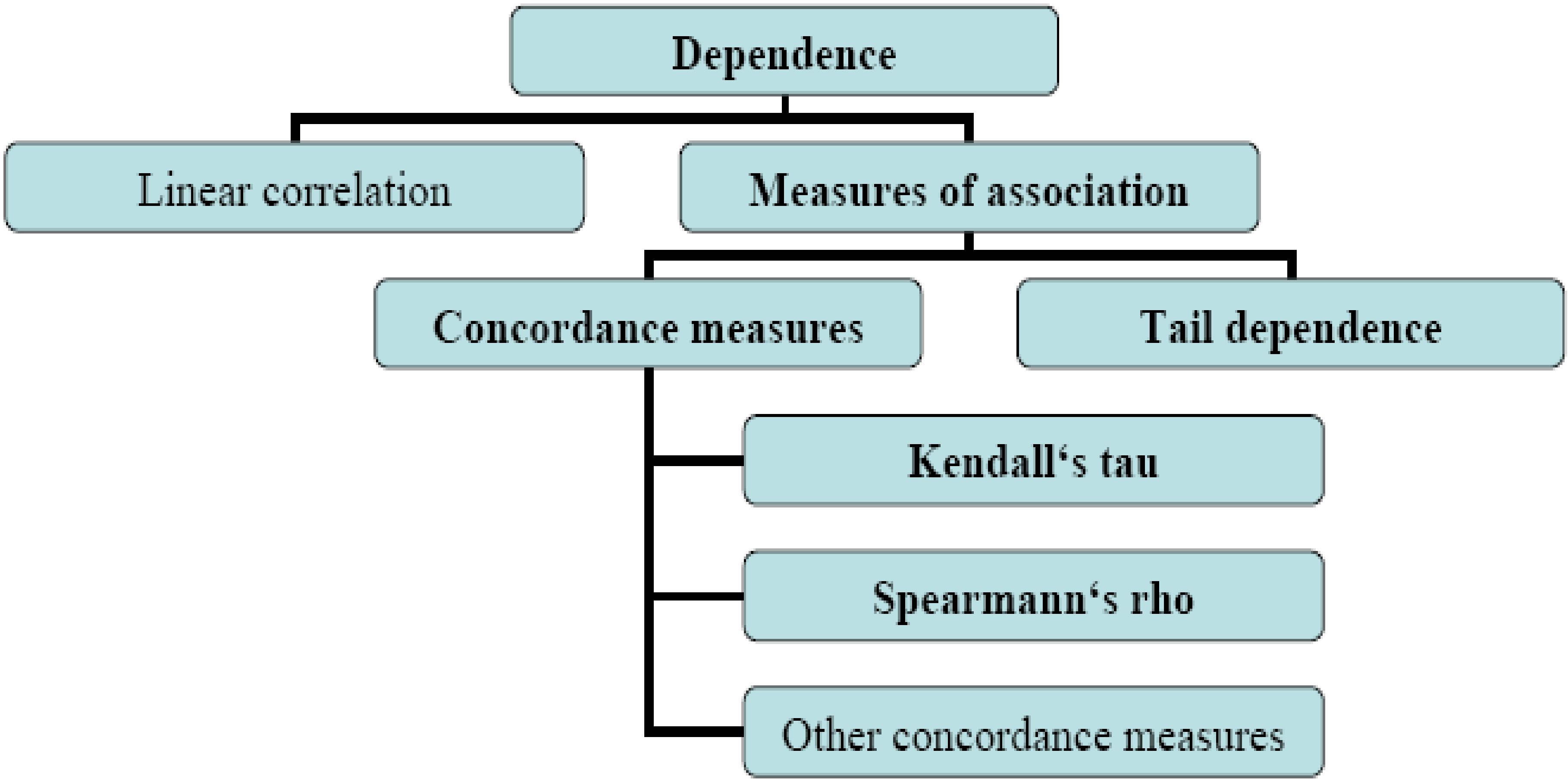
$$\lambda_U = \lim_{u\to 1}\frac{1-2u+C(u,u)}{1-u} = \lim_{u\to 1}\frac{1-2u+\exp\{-[(-\ln u)^{\theta}+(-\ln u)^{\theta}]^{1/\theta}\}}{1-u}$$

$$= \lim_{u\to 1}\frac{1-2u+\exp\{-\{2(-\ln u)^{\theta}]^{1/\theta}\}}{1-u}$$

$$= \lim_{u\to 1}\frac{-2+2^{1/\theta}}{-1} = 2-2^{1/\theta}$$

Dependence

Dependence



Concordance function

$$\hat{\tau}(X,Y) = \frac{\#\text{concordant pairs} - \#\text{discordant pairs}}{\#\text{pairs}}.$$

□ 두 **Data Set(two pairs)**에 대하여 다음과 같이 정의할 수 있음

$$(x_i, y_i) \quad (x_j, y_j)$$

- 일치(concordance) $\quad (x_1 - x_2)(y_1 - y_2) > 0$
- 불일치(disconcordance) $\quad (x_1 - x_2)(y_1 - y_2) < 0$

$$H_1(x,y) = C_1(F(x),G(y)) \quad H_2(x,y) = C_2(F(x),G(y))$$

❑ Concordance function

$$\tau = \Pr[(x_1 - x_2)(y_1 - y_2) > 0] - \Pr[(x_1 - x_2)(y_1 - y_2) < 0]$$

$$= \Pr[(x_1 - x_2)(y_1 - y_2) > 0] - \{1 - \Pr[(x_1 - x_2)(y_1 - y_2) > 0]\}$$

$$= 2\{\Pr[x_1 > x_2, y_1 > y_2] + \Pr[x_1 < x_2, y_1 < y_2]\} - 1$$

❍ Spearman rank correlation

→ **Nonparametric Correlation**

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 \{C(u,v) - uv\}\,du\,dv$$

$$\rho_S(X,Y) = \rho(F_1(X), F_2(Y))$$

$$= \frac{Cov[F_1(X), F_2(Y)]}{\sqrt{\sigma^2[F_1(X)]\sigma^2[F_2(Y)]}} = \frac{Cov[F_1(X), F_2(Y)]}{\sqrt{\frac{1}{12}\frac{1}{12}}}$$

$$= 12Cov[F_1(X), F_2(Y)]$$

$$= 12\int_0^1\int_0^1 \{C(x,y) - F_{F_1(x)}(x)F_{F_2(x)}(y)\}\,dxdy$$

$$= 12\int_0^1\int_0^1 \{C(x,y) - xy\}\,dxdy$$

$$\rho_S(X,Y) = 12\int_0^1\int_0^1 C(u,v)\,du\,dv - 3,$$

$$= 12\int_0^1\int_0^1 C(x,y)\,dxdy - 3$$

$F_1(X), F_2(Y)$ **: Standard uniform distribution**

참고 :

$$Cov[F_1(X), F_2(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F(x,y) - F_1(x)F_2(y)\}\,dxdy$$

❍ Kendall rank correlation

→ **Nonparametric Correlation**

$$\rho_\tau(X,Y) = 4\int_0^1\int_0^1 C(u,v)\,dC(u,v) - 1$$

$$\rho_\tau(X,Y) = 2\Pr[(x_1 - x_2)(y_1 - y_2) > 0] - 1$$

$$= 2E[1_{(x_1-x_2)(y_1-y_2)>0}] - 1$$

$$= 2E[1_{(x_1-x_2)>0}1_{(y_1-y_2)>0} + 1_{(x_1-x_2)<0}1_{(y_1-y_2)<0}] - 1$$

$$= 2[\int\iiint_R 1_{(x_1-x_2)>0}1_{(y_1-y_2)>0}\,dF(x_2,y_2)\,dF(x_1,y_1)$$

$$+\int\iiint_R 1_{(x_1-x_2)<0}1_{(y_1-y_2)<0}\,dF(x_2,y_2)\,dF(x_1,y_1)] - 1$$

$$= 2\square 2\int\iiint_R 1_{(x_1-x_2)>0}1_{(y_1-y_2)>0}\,dF(x_2,y_2)\,dF(x_1,y_1) - 1$$

$$= 4\int_R\int_R F(x_1,y_1)\,dF(x_1,y_1) - 1$$

$$= 4\int_0^1\int_0^1 C(u,v)\,dC(u,v) - 1$$

$$\Pr[(x_1 - x_2)(y_1 - y_2) > 0] + \Pr[(x_1 - x_2)(y_1 - y_2) < 0] = 1$$

Kendall's Tau for Archmedean Copula

$$\tau = 1 + 4\int_0^1 \frac{\varphi(u)}{\varphi'(u)}\,du$$

- **Clayton copula** $$\tau = \frac{\theta}{2+\theta}$$
- **Gumbel copula** $$\tau = 1 - \frac{1}{\theta}$$
- **Frank copula** $$\tau = 1 - 4\,\frac{\int_0^{\theta} \frac{t\,dt}{\exp(t)-1}}{\theta^2}$$

Testing for Elliptic Distributions
– Do we really need to worry about using Copulae at all?

- **Estimating Parametric and Non-parametric Copulae**
  - Maximum Likelihood;
    - Full MLE or sequential / concentrated ML
  - Semi-parametric estimation
    - Impact of the misspecification of marginal distributions on the estimation of copula parameters and measures of dependence
  - Non-parametric methods
- **How to choose the Correct Copula?**
  - AIC
  - Goodness of Fit Tests
  - Simulated Cox Tests
- **Copula Quantile Regression and Tail area Dependence**
  - Quantile conditional relationships implied by a given copula
  - Quantile dependency measures

Gaussian copula의 시뮬레이션

(i) n 개의 상관관계를 갖는 정규변량 $X = (X_1, ...X_n)^T$ 를 상관계수 행렬이 R 인 다변량 표준정규분포로부터 구한다. 이를 위해 다음 알고리즘을 이용한다.

- R 을 Cholesky 분해한 후 A 를 구한다. : $R = AA^T$
- n 개의 서로 독립인 표준정규 확률변량 $z = (z_1, z_2, ..., z_n)^T$ 를 구한다.
- $X = Az$ 로부터 변량 X 를 구한다.

(ii) $u_i = \Phi(X_i)$ 로 변환하여 상관관계를 갖는 [0, 1] 변량 $u = (u_1, ..., u_n)^T$ 를 구한다. $u = (u_1, ..., u_n)^T$ 는 상관계수행렬이 R 인 Gaussian copula로부터 추출한 확률변량이 된다.

(iii) 부도시간 τ_i 의 한계분포의 역함수를 이용하여 $(u_1, ..., u_n)$ 를 부도시간으로 전환하여 개별기업의 부도시간을 구한다.

$$\tau = (\tau_1, \tau_2, ..., \tau_n) = (F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_m^{-1}(u_n))$$

Student's t-copula의 시뮬레이션

(i) n 개의 상관관계를 갖는 t 변량 $X=(X_1,\ldots X_n)^T$ 를 상관계수 행렬과 자유도가 각각 R, ν 인 다변량 t 분포 $t_{R,\nu}(\cdot)$ 로부터 구한다. 이를 위해 다음 알고리즘을 이용한다.

- R 을 Cholesky 분해한 후 A 를 구한다. : $R=AA^T$
- n 개의 서로 독립인 표준정규 확률변량 $z=(z_1,z_2,\ldots,z_n)^T$ 를 구한다.
- x_ν^2 분포로부터 z 와 독립인 확률변량 s 를 구한다
- $y=Az$ 로부터 y 를 구하고, $X=\dfrac{\sqrt{\nu}}{\sqrt{s}}y$ 를 이용 t 분포변량 X 를 구한다.

(ii) $u_i=t_\nu(X_i)$ 로 변환하여 상관관계를 갖는 [0, 1] 변량 $u=(u_1,\ldots,u_n)^T$ 를 구한다. $u=(u_1,\ldots,u_n)^T$ 는 상관계수행렬이 R 이고 자유도가 ν 인 t-copula로부터 추출한 확률변량이 된다.

(iii) 부도시간 τ_i 의 한계분포의 역함수를 이용하여 $(u_1,\ldots,u_n)$ 를 부도시간으로 전환하여 개별기업의 부도시간을 구한다.

$$\tau=(\tau_1,\tau_2,\ldots.,\tau_n)=(F_1^{-1}(u_1),F_2^{-1}(u_2),\ldots\ldots,F_m^{-1}(u_n))$$